# CAUSAL STRUCTURE OF BLACK HOLES



By now, you'll be somewhat familiar with BH metric — for Schwarzschild. Let's do more to unearth the structure of this BH especially in terms of causality.

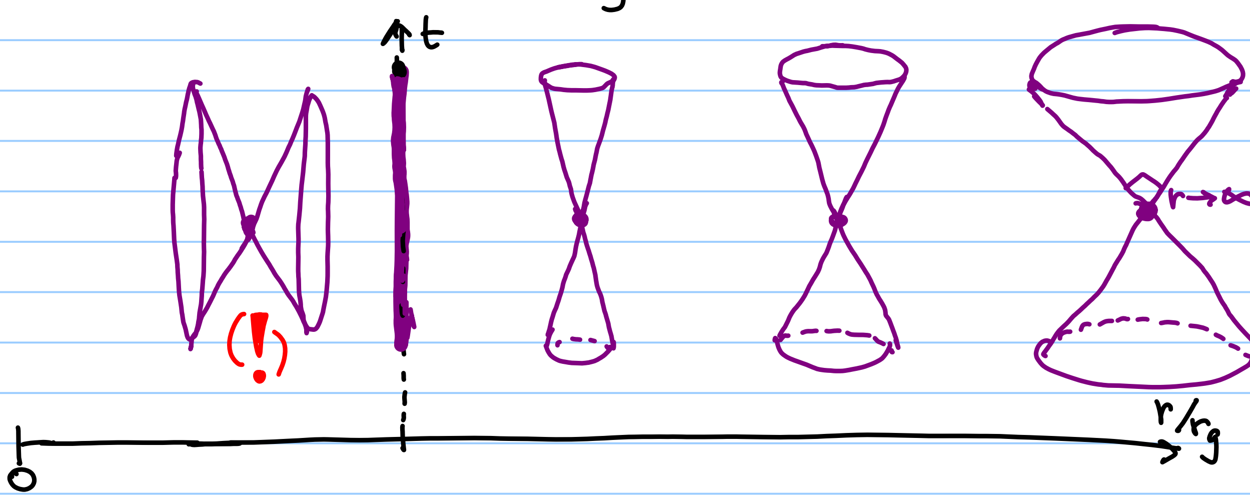
## Light cones

A good place to start is the equation for a null trajectory for purely radial motion:-

(1.1) $$ds^2 = 0 = -\left(1-\frac{r_g}{r}\right)dt^2 + \left(1-\frac{r_g}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2$$

(1.2) $$\Rightarrow \quad \frac{dt}{dr} = \pm\left(1-\frac{r_g}{r}\right)^{-1} \longrightarrow \begin{cases} \pm 1 & \text{at } r \to +\infty \\ \pm\infty & \text{as } r \to r_g \end{cases}$$

So the ticks of $t$ time in this coord system go faster and faster as $r \to r_g$!




But if the photon does not seem to Go anywhere in $(r,\Omega)$ space as $r \to r_g$, does it ever fall into a BH ?!

Actually, yes it does, but horizon-crossing does not occur in these coordinates! – because they break down at the horizon – we had nontrivial $R^\alpha{}_{\beta\gamma\delta}$ (tidal forces) even though $R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}$ is perfectly finite at $r = r_g$ (when $r_g \gg l_p$, that is!)

②

You wouldn't be able to discern this, though, unless you had a decent picture of the geometry and some other coordinate systems to play around with!

(2.1) We've got $\frac{dt}{dr} = \pm \frac{1}{(1 - r_g/r)}$

(2.2) Defining $\frac{dt}{dr_*} = \pm 1$ gives

$$\frac{r_*}{r_g} = \frac{1}{r_g}\int \frac{dr}{(1 - r_g/r)}$$

(2.3) $$\therefore \boxed{\frac{r_*}{r_g} = \frac{r}{r_g} + \ln\left(\frac{r}{r_g} - 1\right)} \qquad \underset{\sim\sim\sim}{r \geq r_g}$$

"tortoise coordinates"

Then $r_* \to \infty$ as $r \to \infty$

and $r_* \to -\infty$ as $r \to r_g$

(2.4) $\Rightarrow$ $r_* \in (-\infty, +\infty)$ covers the region outside $r_g$ only.

In terms of "turtle variables" ☺
we can solve (2.1) as

(2.5) $$\boxed{t = \pm r_* + c.} \text{ (null radial motion)}$$

In these variables

(2.6) $$ds^2 = \left(1 - \frac{r_g}{r}\right)\left(-dt^2 + dr_*^2\right) + r^2(r_*)\, d\Omega_2^2$$

We can next try adapting our coords to null radial motion. Define

(2.7) $$\boxed{\begin{aligned} u &\equiv t - r_* \\ v &\equiv t + r_* \end{aligned}}$$

and then

(2.8) $$\boxed{ds^2 = -\left(1 - \frac{r_g}{r}\right) dv^2 + (dv\,dr + dr\,dv) + r^2\, d\Omega_2^2}$$

Eddington-Finkelstein Coords

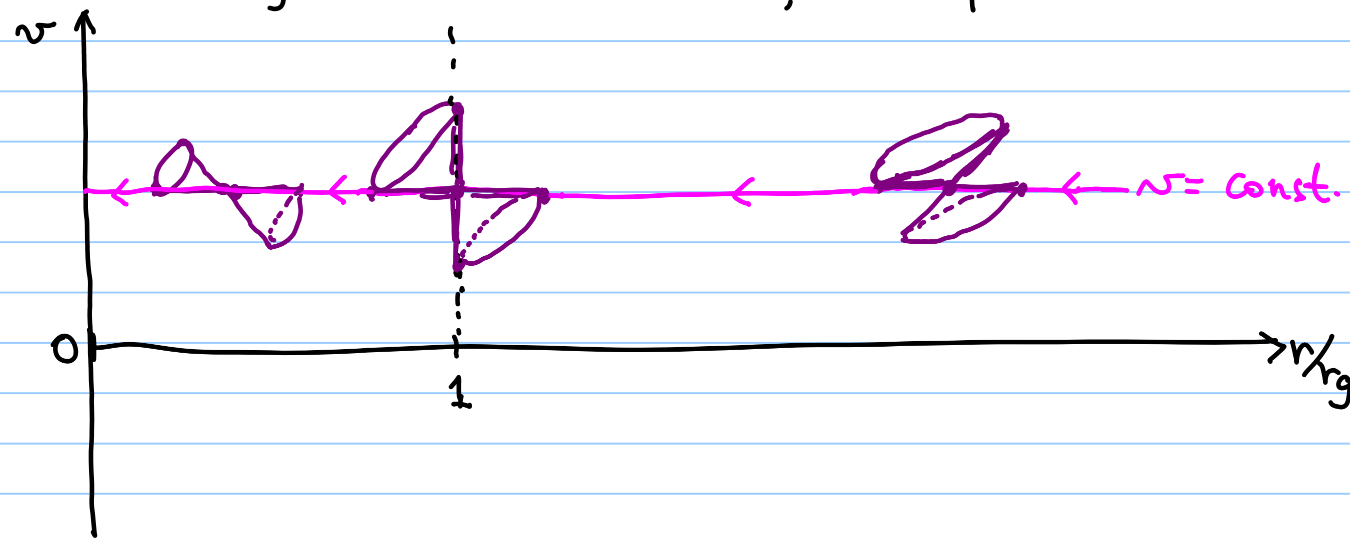
which has det $\neq 0$ at $r = r_g$, unlike $(t, r, \Omega)$ coords.
$(-g)_{EF} = -r^4 \sin^2\theta$

Then for radial null motion the equations can be written

(2.9) in terms of $\frac{dv}{dr} = \frac{dt}{dr} + \frac{dr_*}{dr}$; therefore

(3.1) $$\boxed{\frac{dv}{dr} = \begin{cases} 0\,, & \text{infalling} \\ 2\left(1-\frac{r_g}{r}\right)^{-1}, & \text{outgoing} \end{cases}}$$

In this system of coordinates, the picture is




So the light-cones in Eddington–Finkelstein coords do NOT get squished but they do turn over inside $r=r_g$.

- These E-F coords are nice, but ... are there other coords that restore the "symmetry" between $u$ and $v$?

We could start by finding the coords $(u, r)$ adapted to outgoing motion: we have

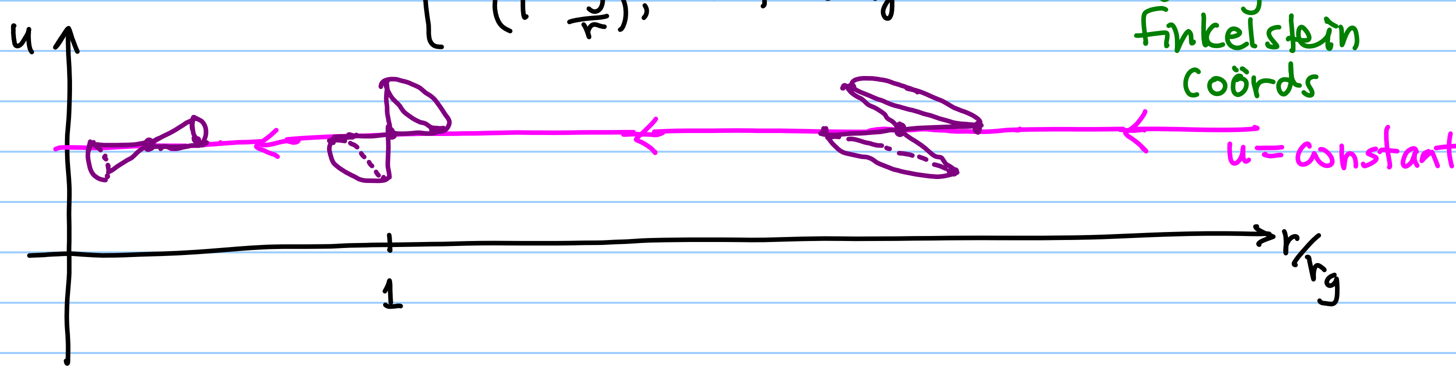
(3.2) $$\boxed{ds^2 = -\left(1-\frac{r_g}{r}\right)du^2 - (du\,dr + dr\,du) + r^2 d\Omega_2^2}$$

Outgoing Eddington-Finkelstein coörds

for which

(3.3) $$\frac{du}{dr} = \begin{cases} 0\,, & \text{outgoing} \\ -2\left(1-\frac{r_g}{r}\right)^{-1}, & \text{infalling} \end{cases}$$

(need to go back in time to escape BH ...)



So ... how about something like a hybrid?
Take

(4.1) $$ds^2 = -\left(1-\frac{r_g}{r}\right)\frac{1}{2}(du\,dv + dv\,du) + r^2(u,v)\,d\Omega_2^2$$

but this was our starting point

(4.2) where $$\frac{r}{r_g} + \ln\left(\frac{r}{r_g}-1\right) = \frac{1}{2}(v-u)$$

But we still have the problem that tortoise coords don't cover any part of spacetime for $r<r_g$ ...
so define

"Pull $r_* = -\infty$ out to finite place"

(4.3) $$\boxed{\tilde{u} = -\exp(-u/2r_g)}$$
$$\boxed{\tilde{v} = +\exp(+v/2r_g)}$$

to avoid that problem. In these new Kruskal coörds we get

(4.4) $$\boxed{ds^2 = \frac{1}{2}(d\tilde{u}\,d\tilde{v} + d\tilde{v}\,d\tilde{u})\left[-\frac{2r_g^3}{r}e^{-r/r_g}\right] + r^2(\tilde{u},\tilde{v})\,d\Omega_2^2}$$

and we can of course make this a tad more familiar-looking by recombining

(4.5a) $$\tilde{t} = \frac{1}{2}(\tilde{u}+\tilde{v}) = \sqrt{r/r_g - 1}\,e^{r/2r_g}\sinh(t/2r_g)$$

(4.5b) $$\tilde{r} = \frac{1}{2}(\tilde{u}-\tilde{v}) = \sqrt{r/r_g - 1}\,e^{r/2r_g}\cosh(t/2r_g)$$

$\Rightarrow$

(4.6) $$\boxed{ds^2 = (-d\tilde{t}^2 + d\tilde{r}^2)\left(-\frac{2r_g^3}{r}e^{-r/r_g}\right) + r^2 d\Omega_2^2}$$

(4.7) with $$\boxed{\tilde{t}^2 - \tilde{r}^2 = \left(1-\frac{r}{r_g}\right)e^{r/r_g}}$$ implicitly defining $r(\tilde{t},\tilde{r})$.

Kruskal[-Szekres] coordinates

Utility

- In Kruskal coords, horizon at
(4.8) $$\tilde{t} = \pm\tilde{r}$$
- Radial null motion along
(4.9) $$\tilde{t} = \pm\tilde{r} + (\text{const.})$$

$\Big\}$ very simple and neat :)

* In Kruskal coords, surfaces of constant $r$ are at
(4.10) $$\tilde{t}^2 - \tilde{r}^2 = \text{constant} \quad \text{(hyperbolae)}$$

Surfaces of constant $t$ are at

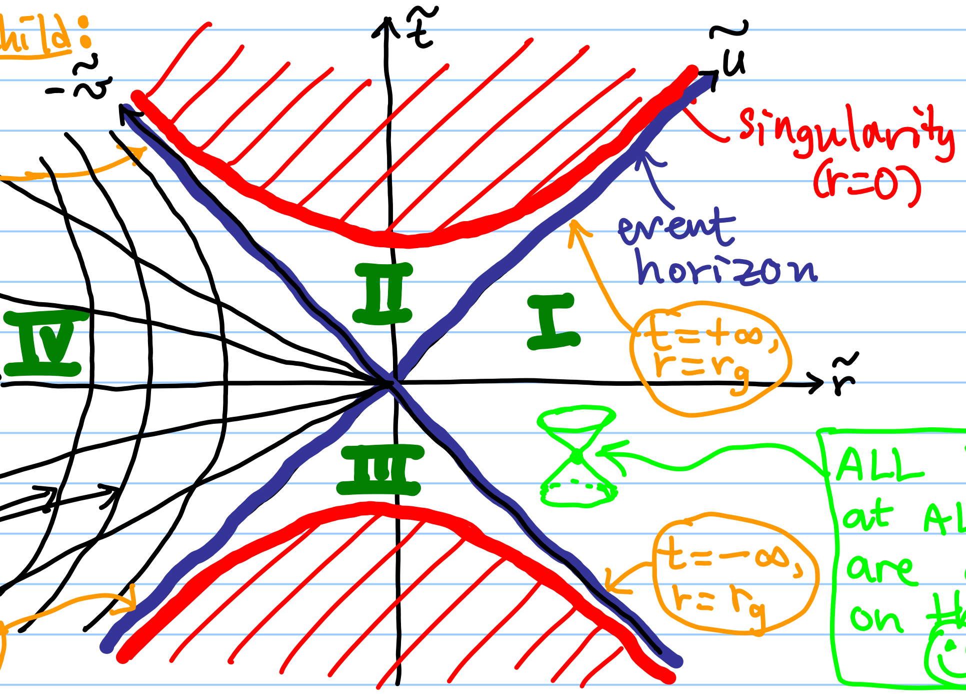
(5.1) $$\frac{\tilde{t}}{\tilde{r}} = \tanh\left(\frac{t}{2r_g}\right)$$

$\Rightarrow$ in the $(\tilde{t}, \tilde{r})$ plane, these are straight lines with slope $\tanh(r/2r_g)$.

▷ Let $(\tilde{t}, \tilde{r})$ range over all possible values aside from where the curvature singularity occurs: i.e. let

(5.2) $$\left\{ \begin{array}{l} -\infty \le \tilde{r} \le +\infty \\ \tilde{t}^2 - \tilde{r}^2 < 1 \end{array} \right\}$$

Maximally Extended Schwarzschild:

(5.3) 


Suppressed $S^2$ at each point; radius $r(\tilde{r})$

- This diagram actually has EXTRA REGIONS by comparison to the original "$r > r_g$ coordinates" $(t, r, \Omega)$! They can be abbreviated II – IV.

- From region I we can, via future-directed null rays, go into region II. So it makes sense to interpret this part as the region behind the black hole event horizon – and you can see from the picture that the singularity is in region II.

- Suppose, from region I, we followed a past-directed null ray. Then what? According to our Kruskal diagram, we would cross a horizon to go into another region – III – with another singularity, the "mirror image" of the singularity in region II. The horizon is also a "mirror image". It is traditional to say that there is a <u>white hole</u>, the time-reverse of a black hole. This has its own horizon .

- By following future-directed null rays from III, or past-directed ones from II, see a second asymptotically flat region! But we can <u>never</u> communicate with it. (Some people talk about Schwarzschild as if it is a "wormhole" connecting asymptotically flat regions, but it isn't physical in any sense to call it a wormhole because it's not traversible: it closes up too quickly for ANY physical observer to cross from I to IV, etc. See spacelike slicings r = const. on p. 228 of Caroll.)

# REISSNER–NORDSTRØM BLACK HOLE #2

⊛ Black hole menagerie is bigger than {Schwarzschild} ☺!

One other famous solution with event horizon is BH solving EM+GR. Suppose started from scratch ?

- Knowing that $A_{\mu,\nu} - A_{\nu,\mu} = A_{\mu;\nu} - A_{\nu;\mu}$ $\boxed{{}_{,\mu} \equiv \partial_\mu}$ $\boxed{{}_{;\mu} \equiv \nabla_\mu}$

because $F = dA$ is antisymmetric

$\Rightarrow$ $F_{\mu\nu}F^{\mu\nu}$ is a bona fide scalar

so

(1.1) $$\boxed{S_{GR+EM} = \frac{1}{16\pi G_D}\int d^D x\sqrt{-g}\left(R - \frac{g_0^2}{4\pi}F^{\mu\nu}F_{\mu\nu}\right)}$$

This has e-o-m

(1.2) $$\boxed{R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = (g_0^2/8\pi)\,T^{(F)}_{\mu\nu}}$$

where

(1.3) $$\boxed{T^{(F)}_{\mu\nu} = \left(F_\mu{}^\lambda F_{\nu\lambda} - \tfrac{1}{4}g_{\mu\nu}F^2\right)}$$

and

(1.4) $$\boxed{\nabla_\mu F^{\mu\nu} = 0}$$

How would we solve this ?

(1.5) Consider $\nabla_\mu F^{\mu\nu} = 0 = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,F^{\mu\nu}\right)$

(1.6) and $d \wedge F_{(2)} = 0$ .

(1.7) d=4, & $\boxed{\text{Spherical symmetric \& static}}$ & $\boxed{\text{Killing vectors } \boxed{\partial_t \,\&\, \partial_\varphi}}$

(1.8) Then (1.6) & (1.7) $\Rightarrow$ $$\boxed{F_{(2)} = F_{tr}(r)\,dt\wedge dr + F_{\theta\varphi}(\theta)\,d\theta\wedge d\varphi}$$

Immediately satisfies $d\wedge F_{(2)} = 0$

Turn to $\nabla_\mu F^{\mu\nu} = 0$ to solve eqns for $F_{tr}$ & $F_{\theta\varphi}$.
Have many ways to approach this. One is via forms.
Find Hodge dual : $*F$ , satisfying $*d*F = 0$ or $d(*F) = 0$

(1.9) And know $(*F)_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}F^{\lambda\sigma}$

(2.1) $$(*F)_{\mu\nu} = \frac{1}{2}\tilde{\epsilon}_{\mu\nu\lambda\sigma}\sqrt{-g}\,g^{\lambda\alpha}g^{\sigma\beta}F_{\alpha\beta}$$

(2.2) Let $\boxed{ds^2 = -e^{2\alpha}dt^2 + e^{2\beta}dr^2 + r^2 d\Omega_2^2}$

(2.3) $$\Rightarrow \sqrt{-g} = e^{(\alpha+\beta)}r^2\sin\theta$$

$$(*F)_{tr} = \tilde{\epsilon}_{tr\theta\varphi}\left(e^{(\alpha+\beta)}r^2\sin\theta\right)\frac{1}{r^2}\frac{1}{r^2\sin^2\theta}F_{\theta\varphi}$$

(2.4) i.e. $\boxed{(*F)_{tr} = \frac{e^{(\alpha+\beta)}}{r^2\sin\theta}F_{\theta\varphi}(\theta)}$

while

$$(*F)_{\theta\varphi} = -\tilde{\epsilon}_{\theta\varphi tr}\left(e^{(\alpha+\beta)}r^2\sin\theta\right)e^{-2\alpha}e^{-2\beta}F_{tr}(r)$$

(2.5) i.e. $\boxed{(*F)_{\theta\varphi} = -e^{-(\alpha+\beta)}r^2\sin\theta\,F_{tr}}$

(2.6) Need $d*F = 0$. Have

$$(*F)_{(2)} = -e^{-(\alpha+\beta)}r^2\sin\theta\,F_{tr}\,d\theta\wedge d\varphi + e^{+(\alpha+\beta)}(r^2\sin\theta)^{-1}F_{\theta\varphi}\,dt\wedge dr$$

$$\Rightarrow d(*F)_{(2)} = -\partial_r\left[e^{-(\alpha+\beta)}r^2F_{tr}(r)\right]\sin\theta\,d\theta\wedge d\varphi + \partial_\theta\left[(\sin\theta)^{-1}F_{\theta\varphi}\right]e^{(\alpha+\beta)}r^{-2}\,dt\wedge dr \equiv 0$$

(2.7) $$\Rightarrow \boxed{F_{tr}(r) = \frac{Q}{4\pi g_0}\frac{e^{(\alpha+\beta)}}{r^2}}$$

(2.8) and $$\boxed{F_{\theta\varphi}(\theta) = \frac{P(\sin\theta)}{4\pi g_0}}$$

← {for the brave at heart (it's a magnetic monopole!!)}

Solving the Einstein equations gives

$$\alpha+\beta=0$$

and thence

(2.9) $$\boxed{ds^2_{RN} = -\Delta\,dt^2 + \Delta^{-1}dr^2 + r^2 d\Omega_2^2}$$

(2.10) $$\boxed{\Delta = 1 - \frac{2GM}{r} + \frac{(Q^2+P^2)}{r^2}}$$

[Carroll sets $\frac{g_0^2}{4\pi} = 16\pi G$ ...]

This has horizons at

$$\boxed{r_\pm = GM \pm \sqrt{G^2M^2 - (Q^2+P^2)}} \tag{3.1}$$

Extremal situation when $r_+ = r_-$

$$\boxed{r_{ex} = GM = +\sqrt{Q^2+P^2}} \tag{3.2}$$

# 5.8 Kerr spacetime

Next to the Schwarzschild spacetime, the Kerr spacetime is the physically most relevant example of a spacetime in which lensing can be studied explicitly in terms of the lightlike geodesics. The Kerr metric is given in Boyer-Lindquist coordinates $(r, \vartheta, \varphi, t)$ as

$$g = \varrho(r,\vartheta)^2 \left( \frac{dr^2}{\Delta(r)} + d\vartheta^2 \right) + (r^2 + a^2) \sin^2 \vartheta \, d\varphi^2 - dt^2 + \frac{2mr}{\varrho(r,\vartheta)^2} \left( a \sin^2 \vartheta \, d\varphi - dt \right)^2 , \qquad (118)$$

where $\varrho$ and $\Delta$ are defined by

$$\varrho(r,\vartheta)^2 = r^2 + a^2 \cos^2 \vartheta, \qquad \Delta(r) = r^2 - 2mr + a^2, \qquad (119)$$

and $m$ and $a$ are two real constants. We assume $0 < a < m$, with the Schwarzschild case $a = 0$ and the extreme Kerr case $a = m$ as limits. Then the Kerr metric describes a rotating uncharged black hole of mass $m$ and specific angular momentum $a$. (The case $a > m$, which describes a naked singularity, will be briefly considered at the end of this section.) The *domain of outer communication* is the region between the (outer) horizon at

$$r_+ = m + \sqrt{m^2 - a^2} \qquad (120)$$

and $r = \infty$. It is joined to the region $r < r_+$ in such a way that past-oriented ingoing lightlike geodesics cannot cross the horizon. Thus, for lensing by a Kerr black hole only the domain of outer communication is of interest unless one wants to study the case of an observer who has fallen into the black hole.

**Historical notes.**
The Kerr metric was found by Kerr [181]. The coordinate representation (118) is due to Boyer and Lindquist [36]. The literature on lightlike (and timelike) geodesics of the Kerr metric is abundant (for an overview of the pre-1979 literature, see Sharp [306]). Detailed accounts on Kerr geodesics can be found in the books by Chandrasekhar [54] and O'Neill [248].

# Kerr-Newman metric

From Wikipedia, the free encyclopedia.

The **Kerr-Newman metric** is a solution of Einstein's general relativity field equation that describes the spacetime geometry around a charged ($Q \neq 0$), rotating ($J \neq 0$) black hole of mass m. The Kerr-Newman metric is:

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)d\phi - adt\right]^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

$$\Delta \equiv r^2 - 2mr + a^2 + Q^2$$

$$\rho^2 \equiv r^2 + a^2\cos^2\theta$$

$$a \equiv \frac{J}{M}$$

The Kerr-Newman metric reduces to the Schwarzschild metric in the uncharged and non-rotating case $Q = a = 0$, the Reissner-Nordstrom metric in the non-rotating case $a = 0$ and the Kerr metric in the uncharged case $Q = 0$. The case $M = Q = 0$ reduces to empty Minkowski space but in an usual spheroidal coordinate system.

As for the Kerr metric, the Kerr-Newman metric defines a black holes only when $a^2 + Q^2 \leq M^2$.

Newman's result represents the most general stationary, axisymmetric solution of Einstein's equations in presence of an electromagnetic field in four dimensions. Although it represents a generalization of the Kerr metric, it is not considered as very important for astrophysical purpose since one does not expect that realistic black holes have an important electric charge.

The Kerr-Newman solution is named after Roy Kerr, discoverer of the uncharged rotating solution named after him (see Kerr metric) and Ezra Ted Newmann, co-discoverer of the charged solution in 1965.

## See also

- Exact solutions in general relativity

## Source(s)


- Kerr-Newman Black Hole (http://scienceworld.wolfram.com/physics/Kerr-NewmanBlackHole.html)

## Killing Horizons

- If any Killing vector becomes null on the horizon, and is null over some hypersurface $\Sigma$, then $\Sigma$ is a Killing horizon

(N.B.: Killing horizons are not necessarily event horizons)

- Classification:
  - Every event horizon $\Sigma$ in a stationary, asymptotically flat spacetime is a Killing horizon for some vector field $\{\chi^\mu\}$
  - For static spacetimes, $\chi^\mu = (\partial_t)^\mu$ (time transl @ $\infty$)
  - For stationary spacetimes, have only axisymmetry and Rotational Killing vector field $R^\mu = (\partial_\phi)^\mu$, and then $\chi^\mu = (\partial_t)^\mu + \Omega_H (\partial_\phi)^\mu$

    $\Omega_H$: angular speed @ horizon

- Having a Killing horizon requires some nontrivial symmetry (i.e. KV(s).)

* Sometimes Killing horizons may not be too interesting
Consider e.g. Minkowski

$$ds^2 = -dt^2 + dx^2 + dy + dz^2$$

Boost K.V. is $\chi = x\partial_t + t\partial_x$ ; norm is $\chi_\mu \chi^\mu = -x^2 + t^2$
goes null @ $x = \pm(c)t$.

More K.H. exist in Minkowski – try repeating the above for other boosts, the rotations, the translations,...

## Surface gravity

Consider a Killing horizon and normal vector to $\Sigma$.
Along KH, $\chi^\mu$ obeys geodesic eqn:

(5.1) $$\boxed{(\chi^\mu \nabla_\mu)\chi^\nu = -\kappa \chi^\nu}$$

(integral curves of $\chi^\nu$ not necessarily affinely parametrized)

- Now make use of Killing equation

(5.2) $$\nabla_{(\mu}\chi_{\nu)} = 0$$

and fact that $\chi \perp \Sigma$ so $\chi_{[\mu}\nabla_\nu \chi_{\sigma]} = 0$

(5.3) $$\boxed{\kappa^2 = -\frac{1}{2}\left[(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)\right]\Big|_{\text{horizon}}}$$

(only) in a static spacetime, $\kappa$ is acceleration of static observer near-horizon, as measured by static observer @ $\infty$.

- For Schwarzschild, suppose $\{\chi^\mu\} = \left\{\frac{\partial}{\partial t}\left(1-\frac{r_g}{r}\right)^{-\frac{1}{2}}, 0, 0, 0\right\}$

$$\kappa^2 = -\frac{1}{2}(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)\Big|_{\text{horizon}}$$

$$= -\frac{1}{2}(\partial_\mu \chi_\nu - \Gamma^\lambda_{\mu\nu}\chi_\lambda)(\partial_\alpha \chi_\beta - \Gamma^\lambda{}_{\alpha\beta}\chi_\lambda) g^{\alpha\mu} g^{\beta\nu}$$

Had, a long time back, nontrivial Christoffels; (see C p.206) we're interested in $\{\Gamma^\mu_{t\nu}\}$

Have $\Gamma^r{}_{tt} = \frac{r_g}{2r^3}(r-r_g)$ ; $\Gamma^t{}_{tr} = \frac{(r_g/2)}{r(r-r_g)}$

and $\{K^\mu\} = \{1, 0, 0, 0\}$

So $-2\kappa^2 = (\nabla_\alpha K^\mu)(\nabla^\alpha K_\mu)$

$= (\Gamma^\mu_{\alpha\beta} K^\beta)(\Gamma^\nu{}_{\gamma\delta} K^\delta) g^{\alpha\gamma} g^{\beta\delta}$

$= -\frac{r_g^2}{4r^4}\Big|_{\text{horizon}} = \frac{-1}{4r_g^2}$

i.e. $\boxed{\kappa^{(S)} = \frac{1}{2r_g}}$ $\Rightarrow$ $\boxed{T_H = \frac{1}{4\pi r_g}}$ ☺

$T dS = dM$ then $\Rightarrow$ $\boxed{S_{BH} = \frac{4\pi r_g^2}{4 G_N}}$ $r_g = GM$ (4d)