(3.1) $$\frac{d^2r}{d\lambda^2} + \frac{r_g}{2r^3}(r-r_g)\left(\frac{dt}{d\lambda}\right)^2 - \frac{r_g}{2r(r-r_g)}\left(\frac{dr}{d\lambda}\right)^2 - (r-r_g)\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{d\lambda}\right)^2\right] = 0$$

and now

$$[\ldots] = 0 + \left(\frac{d\varphi}{d\lambda}\right)^2 = \frac{L^2}{r^4}$$

so that

(3.2) $$\frac{d^2r}{d\lambda^2} + \frac{r_g}{2r^2}\left(1-\frac{r_g}{r}\right)^{-1}E^2 - \frac{r_g}{2r^2}\left(1-\frac{r_g}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - \frac{L^2}{r^3}\left(1-\frac{r_g}{r}\right) = 0$$

This actually has a first integral too, most easily computed by realizing that

(3.3) $$\boxed{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = -\epsilon}$$

must be constant along a geodesic (we varied $\sqrt{\ }$ of this to get the geodesic eqn! :))

For us,

$$-\epsilon = -\left(1-\frac{r_g}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{r_g}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\varphi}{d\lambda}\right)^2$$

$$\epsilon = \left(1-\frac{r_g}{r}\right)^{-1}E^2 - \left(1-\frac{r_g}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - \frac{L^2}{r^2}$$

i.e. $$\left(\frac{dr}{d\lambda}\right)^2 = -\left(\epsilon + \frac{L^2}{r^2}\right)\left(1-\frac{r_g}{r}\right) + E^2 \qquad (\phi)$$

(3.4) $$\Rightarrow \boxed{\frac{dr}{d\lambda} = \mp\sqrt{E^2 - \left(1-\frac{r_g}{r}\right)\left(\epsilon+\frac{L^2}{r^2}\right)}}$$

infalling/outgoing geodesic

▷ Another way to think about this equation is to notice that (ϕ) can be rewritten

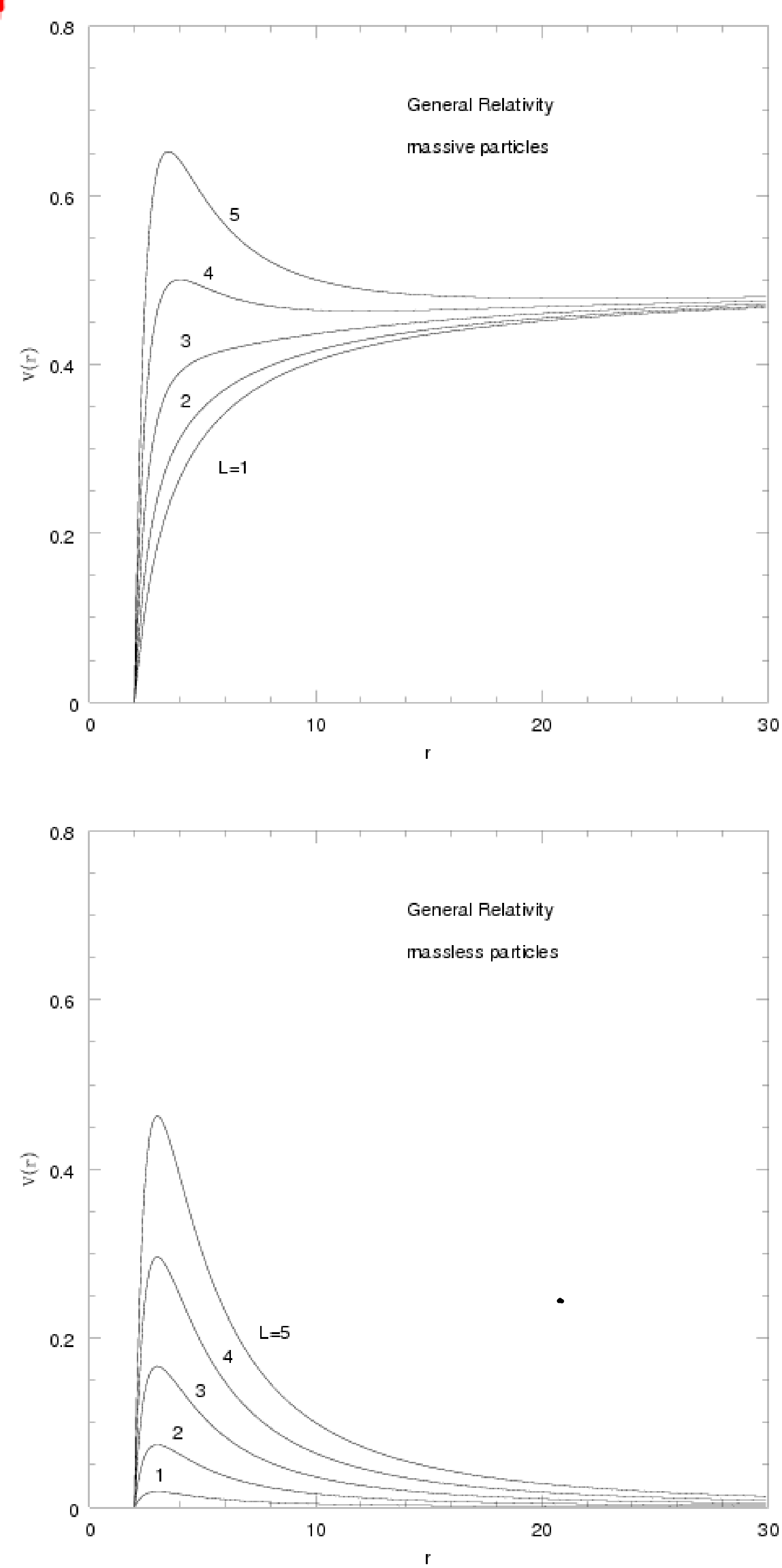
(3.5) $$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V_{eff}(r) = \left(\frac{E^2}{2}\right)$$

(like K.E. + P.E. = total energy = conserved)

⇒ $r(\lambda)$ obeys equation of non-relativistic fame where we imagine λ is non-rel time and

(3.6) $$\boxed{V_{eff} = -\left(1-\frac{r_g}{r}\right)\left(\epsilon+\frac{L^2}{r^2}\right)} = -\epsilon - \frac{L^2}{r^2} + \frac{\epsilon r_g}{r} + \frac{L^2 r_g}{r^3}$$

Picturing $V_{eff}(r(\lambda))$



(*) Please DO ask questions; this is crucial stuff! ☺

Discussion of the physics

The question of main interest is whether there are any turning points for the motion; this will tell us whether

- the probe ("test") particle falls in, or
- the probe can do a circular orbit, or
- the probe misses & goes out to $r \to \infty$ again.

Circular orbit

(5.1) For this, we require $\frac{dV_{eff}}{dr} = 0$

(5.2) Now, $\frac{dV_{eff}}{dr} = \frac{2L^2}{r^3} - \frac{\epsilon r_g}{r^2} - \frac{3L^2 r_g}{r^4}$

$= 0$ at $r = r_c$

$$\Rightarrow \epsilon r_g r_c^2 - 2L^2 r_c + 3L^2 r_g = 0$$

So $r_c = \left(+2L^2 \pm \sqrt{4L^4 - 12L^2 \epsilon r_g^2}\right) \frac{1}{2\epsilon r_g}$

(5.3) $= \frac{L^2}{\epsilon r_g} \pm \frac{L^2}{\epsilon r_g}\sqrt{1 - 3\frac{\epsilon r_g^2}{L^2}}$ when $\epsilon \neq 0$ (*)

and when $\epsilon = 0$ we have

(5.4) $\boxed{r_{c(ICO)} = \frac{3}{2} r_g = 3GM}$ Innermost circular orbit

Expanding (*) for small-ϵ we have

$r_c \simeq \frac{L^2}{\epsilon r_g}\left[1 \pm \left(1 - \frac{3}{2}\frac{\epsilon r_g^2}{L^2}\right) + \mathcal{O}(\epsilon^2)\right]$

(5.6) $= \frac{L^2}{\epsilon r_g}(1 \pm 1) \mp \frac{3}{2} r_g \Rightarrow \ominus$ sqrt sign choice

(5.7) $\Rightarrow \boxed{r_c = \frac{L^2}{\epsilon r_g}\left[1 \underset{+}{-} \sqrt{1 - 3\frac{\epsilon r_g^2}{L^2}}\right]}$ (small-ish L^2) (‡) larger L^2

For $m^2 > 0$ & $L^2 \gg 1$, there are actually two solutions which lie outside r_{c*}, given by

(5.8) $r_{c,1} \simeq \frac{L^2}{GM}$ and $r_{c,2} \simeq 3GM^{+}$

(6.1) N.B.: $\epsilon = 0$ for massless particles
$\Rightarrow$ photons can orbit forever at $r_c = r_{c*}$ (ONLY!)
Any photon moving a bit in or out must either fall into the black hole or escape to ∞.
It may buzz around the BH (outside of $r = r_g$) a few times before flitting off to $r \to \infty$ ☺

(6.2) (‡) In fact, for $L^2 \gg 1$ this implies that the inner orbit is the unstable one, as it matches with the photonic circular orbit.
$\Rightarrow$ (*) the stable orbit is at larger radius! ($L^2 \gg 1$)

- When do the stable & unstable orbits coalesce?

(6.3) When $\sqrt{1 - \frac{3\epsilon r_g^2}{L^2}} = 0$

(N.B.: impossible for photons)

(6.4) $\Rightarrow$ $L^2 = 3\epsilon r_g^2$ so that

(6.5) $$\boxed{r_{c,\mathrm{ISCO}} = 3 r_g} = 6GM = 2 r_{c,\mathrm{ICO}}$$

(P.S. All of this is analyzed in GR.
If there were an "alternative" theory of gravity, perhaps involving a scalar-tensor story and/or other terms in $\mathcal{L}$, and/or we had different coupling of $g_{\mu\nu}$ to matter fields than we do, then these calculations would need to be redone.)

Precession of Perihelion

- By Birkhoff's theorem the exterior Schwarzschild metric is very important for astrophysical applications. (Great! ☺)
- Orbits in GR actually do not follow conic sections; they are approximately ellipses that precess.
- Computed geodesic equation. Combining $r(\lambda)$ eqn and $\phi(\lambda)$ eqns gives

(1.1) $$\left(\frac{dr}{d\phi}\right)^2 + \frac{r^4}{L^2}\left(1-\frac{r_g}{r}\right) + r^2\left(1-\frac{r_g}{r}\right) = \frac{2\mathcal{E}}{L^2}r^4 \qquad \left(\mathcal{E} \equiv \tfrac{1}{2}E^2\right)$$

(1.2) Define $x \equiv \frac{2L^2}{r_g r}$ (useful abbreviation).

(1.3) Then $$\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{2L}{r_g}\right)^2 - 2x + x^2 - \frac{r_g^2}{2L^2}x^3 = \frac{\mathcal{E}L^2}{2r_g^2}$$

so $$\left(\frac{dx}{d\phi}\right)^2 + \left[\left(\frac{2L}{r_g}\right)^2 - \frac{\mathcal{E}L^2}{2r_g^2}\right] - 2x + x^2 - \frac{r_g^2}{2L^2}x^3 = 0$$

Acting on this with $(d/d\phi)$ gives
$$2\left(\frac{d^2x}{d\phi^2}\right)\cdot\left(\frac{dx}{d\phi}\right) - 2\left(\frac{dx}{d\phi}\right) + 2x\left(\frac{dx}{d\phi}\right) - \frac{3r_g}{2L^2}x^2\frac{dx}{d\phi} = 0$$

Cancelling a common factor of $2(dx/d\phi)$ (whew! ☺) gives

(1.4) $$\boxed{\frac{d^2x}{d\phi^2} - 1 + x = \boxed{3\left(\frac{r_g}{2L}\right)^2}x^2}$$ ← absent for old Newtonian story (cf.)

- Expand $x = x_0 + x_1$

(1.5a) Newtonian
(1.5b) piece $$\boxed{\frac{d^2x_0}{d\phi^2} = 1 - x_0}$$ $$\boxed{\frac{d^2x_1}{d\phi^2} + x_1 = \underbrace{3\left(\frac{r_g}{2L}\right)^2 x_0^2}_{\text{"source" for } x_1}}$$ Perturbation piece

(1.6a) Solution $\boxed{x_0 = 1 + e\cos\phi}$

ellipse ; $e^2 = 1-(b/a)^2$

Math ((Carroll p.215) ⇒

(1.6b) $$\boxed{x_1 = 3\left(\frac{r_g}{2L}\right)^2\left[\underbrace{\left(1+\frac{e^2}{2}\right)}_{\text{constant displacement}} + e\phi\sin\phi - \underbrace{\frac{e^2}{6}\cos 2\phi}_{\text{oscillates about }0}\right]}$$

Consider $x = 1 + e\phi\cos\phi + 3\left(\frac{r_g}{2L}\right)^2 e\phi\sin\phi$ [x_0 plus 2nd term only in x_1]

$$\cong 1 + e\cos[(1-\alpha)\phi], \quad \alpha = 3(r_g/2L)^2$$

(1.7) where $$\boxed{\Delta\phi = 2\pi\alpha = \frac{6\pi G^2M^2}{L^2c^4}}$$ cf. famed precession of perihelion of Mercury:– 43″ per century

(1.8) and ex x_0 $$\boxed{L^2 \cong GM(1-e^2)a}$$

CAUSAL STRUCTURE OF BLACK HOLES

By now, you'll be somewhat familiar with BH metric - for Schwarzschild. Let's do more to unearth the structure of this BH especially in terms of causality.

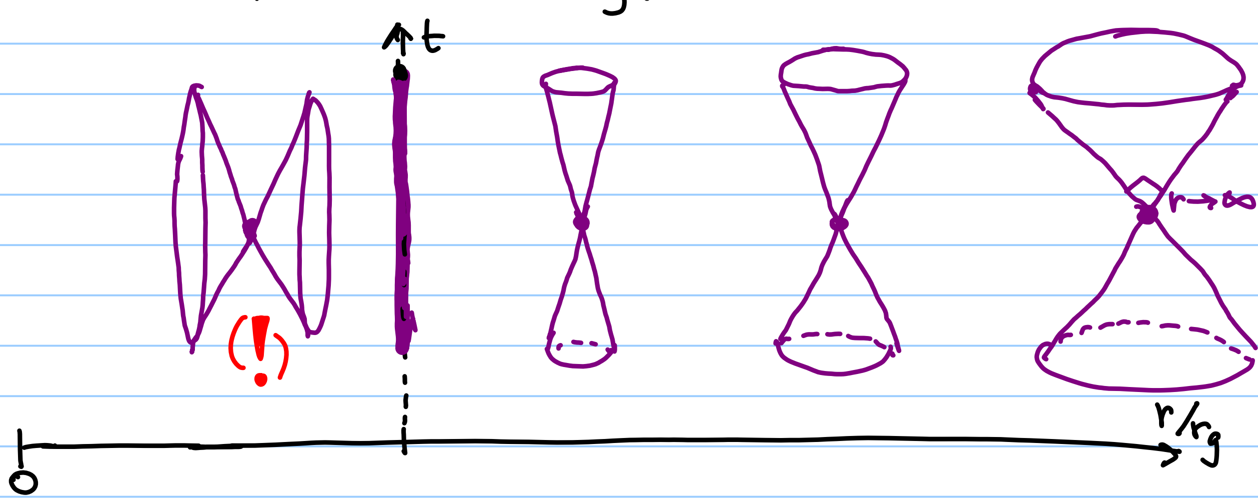
Light cones

A good place to start is the equation for a null trajectory for purely radial motion :-

(1.1) $$ds^2 = 0 = -\left(1-\frac{r_g}{r}\right) dt^2 + \left(1-\frac{r_g}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

(1.2) $$\Rightarrow \quad \frac{dt}{dr} = \pm\left(1-\frac{r_g}{r}\right)^{-1} \longrightarrow \begin{cases} \pm 1 & \text{at } r\to+\infty \\ \pm\infty & \text{as } r\to r_g \end{cases}$$

So the ticks of t time in this coord system go faster and faster as $r\to r_g$!



But if the photon does not seem to Go anywhere in (r,Ω) space as $r\to r_g$, does it ever fall into a BH ?!

Actually, yes it does, but horizon-crossing does not occur in these coordinates! - because they break down at the horizon - we had nontrivial $R^\alpha{}_{\beta\gamma\delta}$ (tidal forces) even though $R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}$ is perfectly finite at $r=r_g$ (when $r_g \gg l_p$, that is!)

You wouldn't be able to discern this, though, unless you had a decent picture of the geometry and some other coordinate systems to play around with!

(2.1) We've got $\frac{dt}{dr} = \pm \frac{1}{(1-r_g/r)}$

(2.2) Defining $\frac{dt}{dr_*} = \pm 1$ gives

$$\frac{r_*}{r_g} = \frac{1}{r_g}\int \frac{dr}{(1-r_g/r)}$$

(2.3) $$\therefore \boxed{\frac{r_*}{r_g} = \frac{r}{r_g} + \ln\left(\frac{r}{r_g}-1\right)} \qquad r \geq r_g$$

"tortoise coordinates"

Then $r_* \to \infty$ as $r \to \infty$

and $r_* \to -\infty$ as $r \to r_g$

(2.4) $\Rightarrow$ $r_* \in (-\infty, +\infty)$ covers the region <u>outside</u> r_g <u>only</u>.

In terms of "turtle variables" ☺ we can solve (2.1) as

(2.5) $$\boxed{t = \pm r_* + c.} \text{ (null radial motion)}$$

In these variables

(2.6) $$ds^2 = \left(1-\frac{r_g}{r}\right)\left(-dt^2 + dr_*^2\right) + r^2(r_*)\, d\Omega_2^2$$

We can next try adapting our coords to null radial motion. Define

(2.7) $$\boxed{\begin{aligned} u &\equiv t - r_* \\ v &\equiv t + r_* \end{aligned}}$$

and then

Eddington–Finkelstein Coords

(2.8) $$\boxed{ds^2 = -\left(1-\frac{r_g}{r}\right)dv^2 + (dv\,dr + dr\,dv) + r^2\, d\Omega_2^2}$$

which has det $\neq 0$ at $r=r_g$, unlike (t,r,Ω) coords.

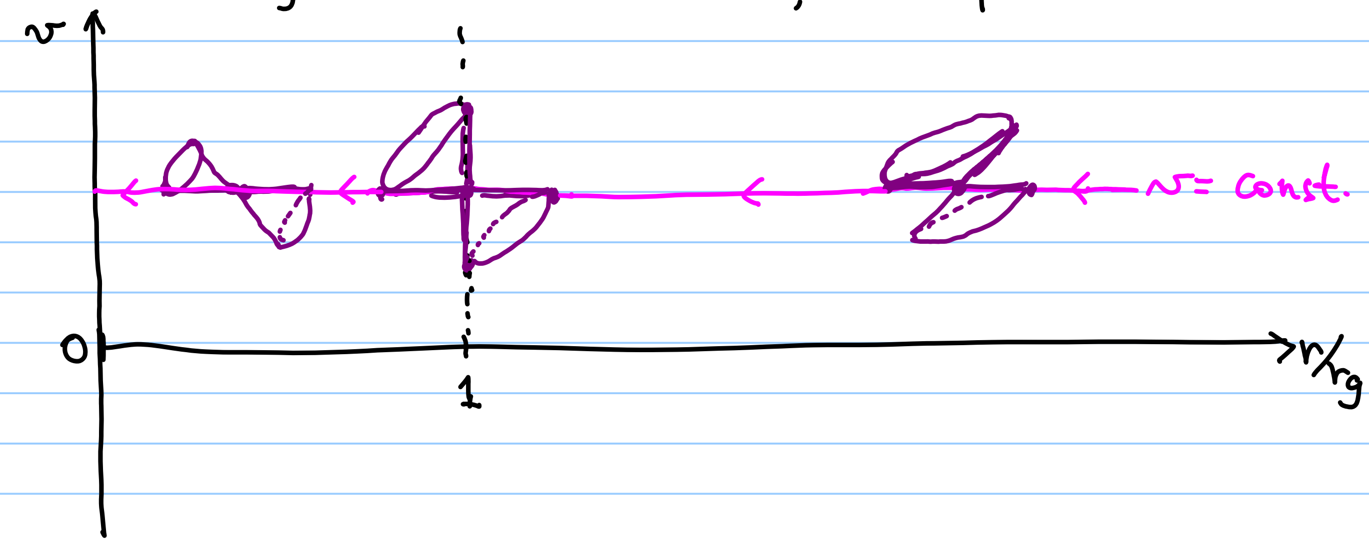
$(-g)_{EF} = -r^4 \sin^2\theta$

Then for radial null motion the equations can be written

(2.9) in terms of $\frac{dv}{dr} = \frac{dt}{dr} + \frac{dr_*}{dr}$; therefore

(3.1) $$\boxed{\frac{dv}{dr} = \begin{cases} 0\,, & \text{infalling} \\ 2\left(1-\frac{r_g}{r}\right)^{-1}, & \text{outgoing} \end{cases}}$$

In this system of coordinates, the picture is



So the light-cones in Eddington–Finkelstein coords do NOT get squished but they do turn over inside $r = r_g$.

- These E-F coords are nice, but ... are there other coords that restore the "symmetry" between u and v?

We could start by finding the coords (u, r) adapted to outgoing motion: we have

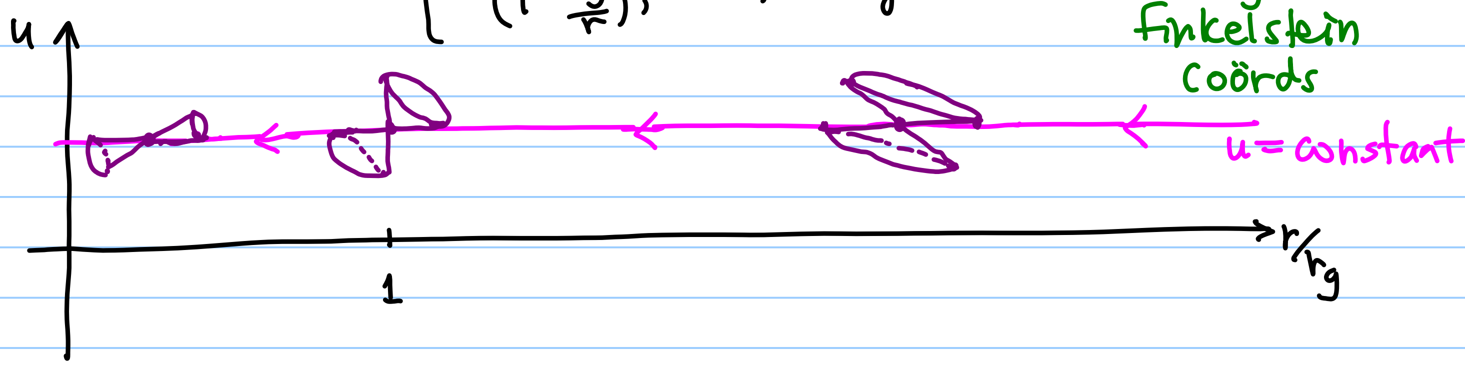
(3.2) $$\boxed{ds^2 = -\left(1-\frac{r_g}{r}\right) du^2 - (du\,dr + dr\,du) + r^2 d\Omega_2^2}$$

for which

(3.3) $$\frac{du}{dr} = \begin{cases} 0\,, & \text{outgoing} \\ -2\left(1-\frac{r_g}{r}\right)^{-1}, & \text{infalling} \end{cases}$$

(need to go back in time to escape BH...)

Outgoing Eddington-Finkelstein coörds



So ... how about something like a hybrid?
Take

(4.1) $$ds^2 = -\left(1-\frac{r_g}{r}\right)\frac{1}{2}(du\,dv + dv\,du) + r^2(u,v)\,d\Omega_2^2$$

but ~~this~~ was • our starting point

(4.2) where $$\frac{r}{r_g} + \ln\left(\frac{r}{r_g}-1\right) = \frac{1}{2}(v-u)$$

But we still have the problem that tortoise coords don't cover any part of spacetime for $r<r_g$...
so define

"Pull $r_* = -\infty$ out to finite place"

(4.3) $$\boxed{\tilde u = -\exp(-u/2r_g)} \qquad \boxed{\tilde v = +\exp(+v/2r_g)}$$

to avoid that problem. In these new Kruskal coörds we get

(4.4) $$\boxed{ds^2 = -\frac{1}{2}(d\tilde u\, d\tilde v + d\tilde v\, d\tilde u)\left[-\frac{2r_g^3}{r}e^{-r/r_g}\right] + r^2(\tilde u,\tilde v)\,d\Omega_2^2}$$

and we can of course make this a tad more familiar-looking by recombining

(4.5a) $$\tilde t = \frac{1}{2}(\tilde u + \tilde v) = \sqrt{r/r_g - 1}\; e^{r/2r_g}\sinh(t/2r_g)$$

(4.5b) $$\tilde r = \frac{1}{2}(\tilde u - \tilde v) = \sqrt{r/r_g - 1}\; e^{r/2r_g}\cosh(t/2r_g)$$

$\Rightarrow$

(4.6) $$\boxed{ds^2 = (-d\tilde t^2 + d\tilde r^2)\left(-\frac{2r_g^3}{r}e^{-r/r_g}\right) + r^2 d\Omega_2^2}$$

(4.7) with $$\boxed{\tilde t^2 - \tilde r^2 = \left(1-\frac{r}{r_g}\right)e^{r/r_g}}$$ implicitly defining $r(\tilde t,\tilde r)$.

Kruskal[-Szekres] coordinates

Utility

- In Kruskal coords, horizon at

(4.8) $$\tilde t = \pm\tilde r$$

- Radial null motion along

(4.9) $$\tilde t = \pm\tilde r + (\text{const.})$$

} very simple and neat :)

* In Kruskal coords, surfaces of constant r are at

(4.10) $$\tilde t^2 - \tilde r^2 = \text{constant} \quad \text{(hyperbolae)}$$

Surfaces of constant t are at

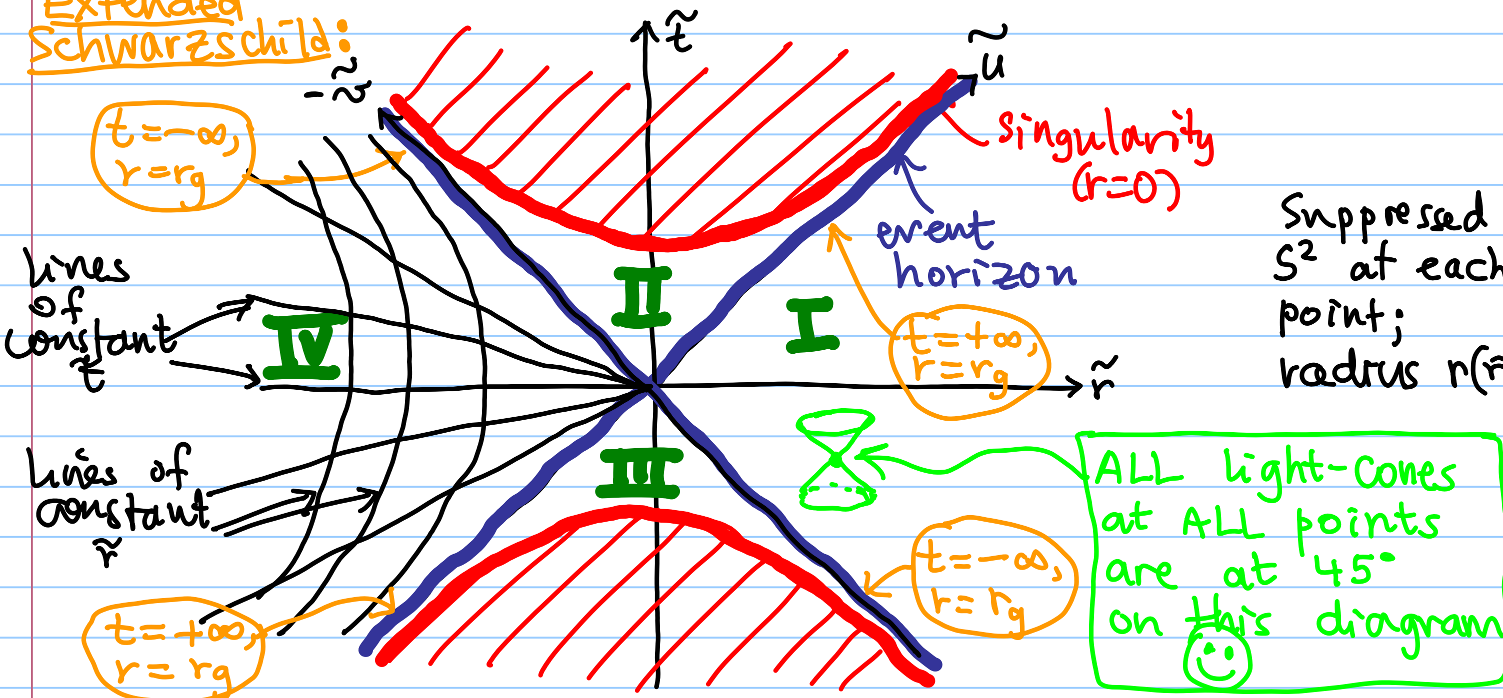
(5.1) $$\frac{\tilde{t}}{\tilde{r}} = \tanh\left(\frac{t}{2r_g}\right)$$

$\Rightarrow$ in the $(\tilde{t},\tilde{r})$ plane, these are straight lines with slope $\tanh(r/2r_g)$.

▷ Let $(\tilde{t},\tilde{r})$ range over all possible values aside from where the curvature singularity occurs: i.e. let

(5.2) $$\left\{ \begin{array}{l} -\infty \leq \tilde{r} \leq +\infty \\ \tilde{t}^2 - \tilde{r}^2 < 1 \end{array} \right\}$$

Maximally Extended Schwarzschild:

(5.3) 

Suppressed S^2 at each point; radius $r(\tilde{r})$

- This diagram actually has EXTRA REGIONS by comparison to the original "$r > r_g$ coordinates" (t, r, Ω)! They can be abbreviated II – IV.

- From region I we can, via future-directed null rays, go into region II. So it makes sense to interpret this part as the region behind the black hole event horizon – and you can see from the picture that the singularity ▬ is in region II.

- Suppose, from region I, we followed a past-directed null ray. Then what? According to our Kruskal diagram, we would cross a horizon to go into another region - III - with another singularity, the "mirror image" of the singularity in region II. The horizon is also a "mirror image". It is traditional to say that there is a white hole, the time-reverse of a black hole. This has its own horizon ▬.

- By following future-directed null rays from III, or past-directed ones from II, see a second asymptotically flat region! But we can never communicate with it. (Some people talk about Schwarzschild as if it is a "wormhole" connecting asymptotically flat regions, but it isn't physical in any sense to call it a wormhole because it's not traversible: it closes up too quickly for ANY physical observer to cross from I to IV, etc. See spacelike slicings r = const. on p.228 of Carroll.)

Next time.

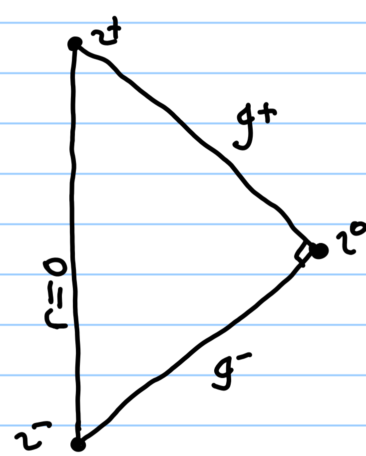
Conformal diagrams ("Carter-Penrose diagrams") (Appendix H)

There is a really cool idea that helps heaps when trying to picture a spacetime and especially its causal structure. Useful for grad+ research.

The basic idea is to bring infinity to a finite place so it can be drawn on the page, and to have null rays go at 45° in the (time, radius) plane. Suppress transverse S^2.

Start with Minkowski space-time.

Next lecture, I will show how to construct its Penrose diagram. The result will look like this ⟶



Penrose Diagrams

Consider (t, r) part of a spacetime i.e., suppress $\perp S^2$. Keep it in mind; r^2 in front of $d\Omega_2^2$ describes [varying] sphere radius.

▷ Aim: portray causal structure

- (radial) Light cones are always at $45°$ on the diagram
- apply a conformal transformation $\tilde{g}_{\mu\nu} = \omega^2(x)\, g_{\mu\nu}$ to bring infinity in to finite place.

Minkowski space

(2.1) $ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2$ $\qquad (-\infty < t < +\infty \ , \ 0 \le r < \infty)$

(2.2) Trajectories $t = \pm r$ are null.

Switch to null coords

(2.3) $$\begin{Bmatrix} u = t - r \\ v = t + r \end{Bmatrix}$$

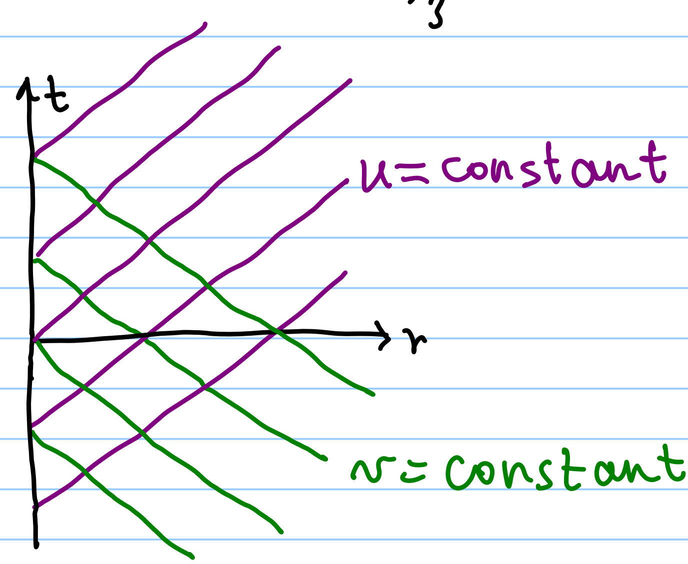
then

(2.4) $$ds^2 = -du\,dv + r^2(u,v)\, d\Omega_2^2 \qquad \left[\text{"}\tfrac{1}{2}(du\,dv + dv\,du)\text{"}\right] \text{ in Carroll...}$$

(2.5) where $\begin{Bmatrix} -\infty < u < +\infty, \ -\infty < v < +\infty; \\ u \le v \end{Bmatrix}$

On a diagram,

(2.6) →



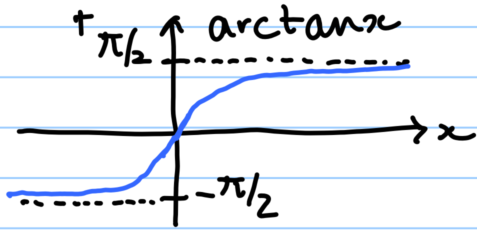
We can use a trick first met last lecture:

(3.1) $$\left\{\begin{array}{l} U = \arctan(u) \\ V = \arctan(v) \end{array}\right\}$$ Brings ∞ in to a finite place

hence

(3.2) $$-\pi/2 < U \leq +\pi/2 \,, \quad -\pi/2 < V < +\pi/2 \;; \quad U \leq V$$

(3.3) (



)

(3.4) Using $g_{\mu'\nu'} = \frac{\partial x^\lambda}{\partial x^{\mu'}} \frac{\partial x^\sigma}{\partial x^{\nu'}} g_{\lambda\sigma}$, find straightforwardly that

(3.5) $$ds^2 = \frac{1}{4\cos^2 U \cos^2 V}\left[-4\,dU\,dV + \sin^2(V-U)\,d\Omega_2^2\right]$$

or in

(3.6) $$\left\{\begin{array}{l} T = (U+V) \\ R = (-U+V) \end{array}\right\}$$

(3.7) $$\boxed{ds^2 = \frac{1}{\omega^2(T,R)}\left[-dT^2 + dR^2 + \sin^2 R\,d\Omega_2^2\right]}$$

where

$$\omega(T,R) = 2\cos\left[\tfrac{1}{2}(T-R)\right]\cos\left[\tfrac{1}{2}(T+R)\right]$$

(3.8) i.e. $$\boxed{\omega(T,R) = \cos T + \cos R}$$

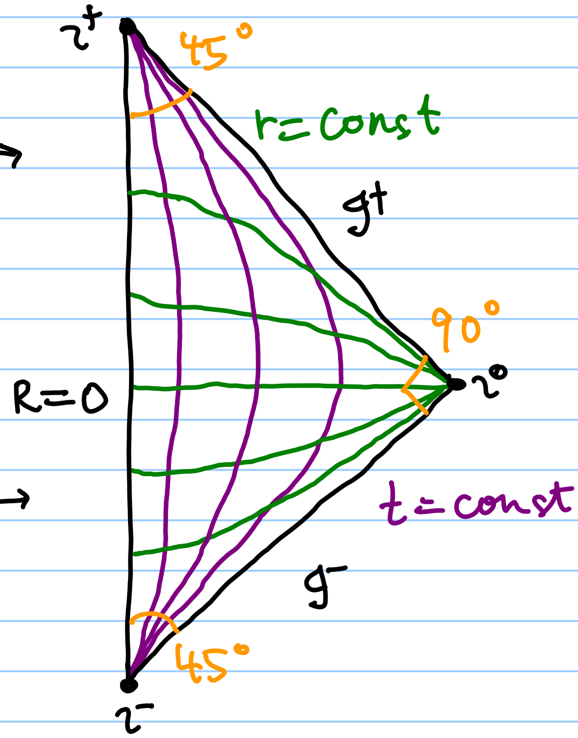
Conformal transformation of metric $\tilde{g}$ on $\mathbb{R}\times S^3$ [...].

- Curvature of S^3 not important: metric on $\mathbb{R}\times S^3$ is NOT physical; this conformally transformed metric is just auxiliary. Note also that null trajectories in radial direction are the same for $\tilde{g}$ as for g (!)
  $\Rightarrow$ extract causal structure from $\tilde{g}$ which is simpler. ☺

↓

Penrose Diagram of Minkowski

(4.1) →



(4.2) →

Light cone at all points: 45°.

- i^+ = future timelike ∞
- i^- = past timelike ∞
- i^0 = spatial ∞
- $\mathscr{I}^-$ = past null ∞
- $\mathscr{I}^+$ = future null ∞

Penrose for "$a \sim t^q$ FRW"

Suppose consider

(4.3) $$ds^2 = -dt^2 + t^{2q}(dr^2 + r^2 d\Omega_2^2)$$

(4.4) or with $d\eta = t^{-q}\, dq$ (i.e. $\eta = \frac{1}{(1-q)} t^{1-q}$)

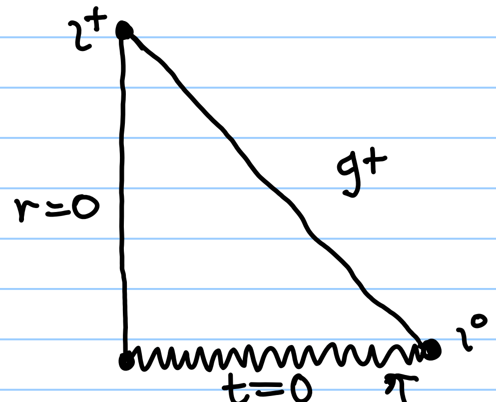
we have

(4.5) $$ds^2 = t^{2q}(-dt^2 + dr^2 + r^2 d\Omega_2^2)$$

↑ ω^{-2} times flat metric with conformal factor $t^{-q} \to \infty$ as $t \to 0$

⇒

(4.6) →

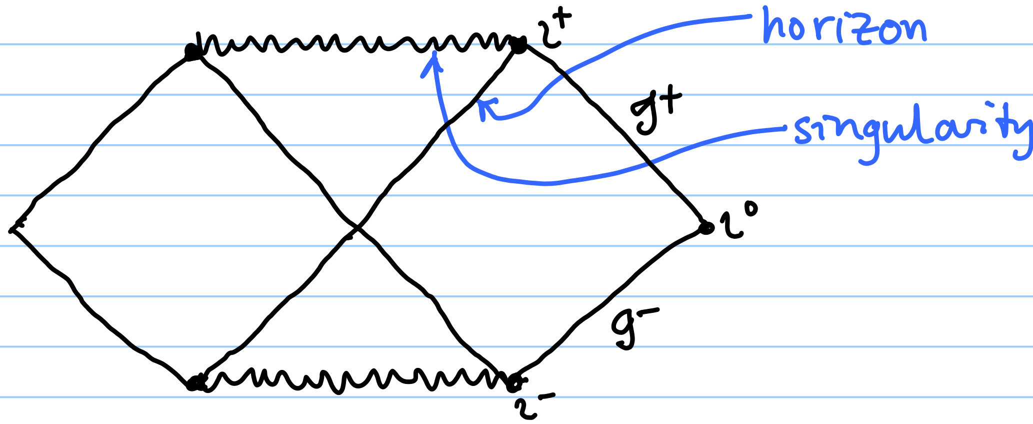


Spacetime simply <u>ends</u> as $t \to 0^+$.

Schwarzschild : eternal BH

Time-symmetric BH & WH

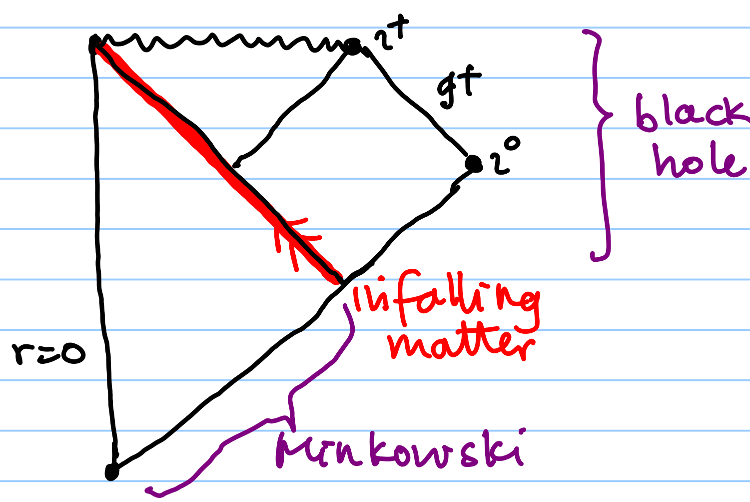
(5.1) →



Schwarzschild : gravitational collapse

Simplest collapse : null matter. Then paste on Minkowski space. ⇒

(5.2) →



Not time-symmetric - because gravitational collapse is not time-symmetric.

- Note: horizon begins after matter falls inside its own Schwarzschild radius.

(5.3) ⊛ $r_g = \frac{2 G_N M}{c^2} > l_p$ for $m > m_p \sim 10^{-5}$ g