Curvature Singularity

- Some drama queens insist that Einstein's theory was "doomed" from the beginning, because it "contained the seeds of its own destruction"! I prefer to say that

⊛ Einstein's theory of gravity, our friend GR, works well as an effective theory – ONLY when <u>all curvature invariants</u> are <u>small</u>, <u>in Planck units</u>.

(4.1) Require $$\boxed{|R|\, \ell_P^2 \ll 1\ , \quad |R^{\mu\nu}R_{\mu\nu}|\, \ell_P^4 \ll 1, \quad |R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}|\, \ell_P^4 \ll 1, \ \ldots}$$ ← grtensor v. useful for this

These are statements we can all agree upon, because they are all <u>dimensionless</u>, <u>scalar</u> statements. ☺

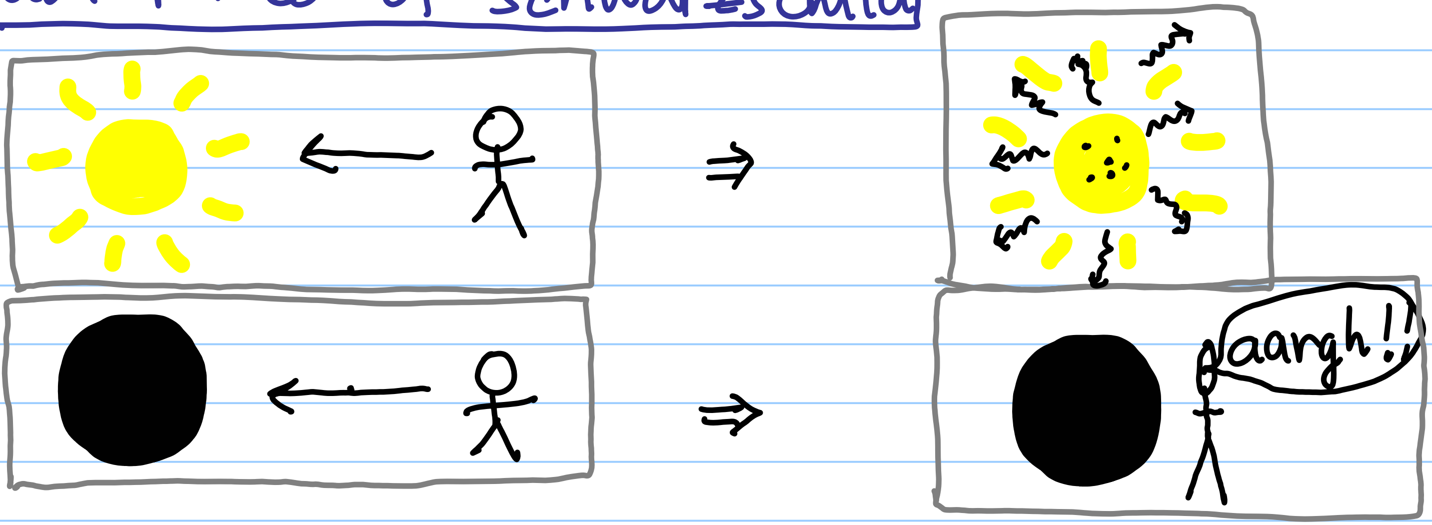
(4.2) • Singularity of Schwarzschild is at the core: $\boxed{r = 0}$

(4.3) where, for example, $\boxed{R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} = -\dfrac{12 r_g^2}{r^6}}$

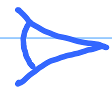
(4.4) while $\underbrace{R \equiv 0 \text{ and } R_{\mu\nu} \equiv 0}$ (!)

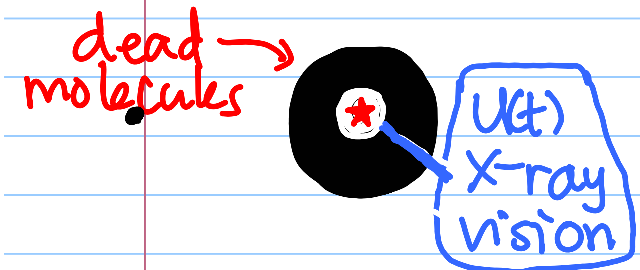
[This follows from $T_{\mu\nu} \Rightarrow G_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R$
so, tracing this gives $R = \frac{1}{2}(D) R$ i.e. $R = 0$ (unless $D = 2$)
Substituting $R = 0$ gives $R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} (\emptyset) \equiv 0$. ∎]

Tidal Forces of Schwarzschild



(whoops! ☺)





- Astronauts falling in to stars get fried;
- those going into black holes first get hung, drawn and quartered by <u>tidal forces</u>.

- Recall separation vector S^μ of neighbour geodesics:

$$A^\mu = \frac{D}{d\tau} V^\mu = \boxed{\frac{D^2 S^\mu}{d\tau^2} = R^\mu{}_{\nu\lambda\sigma} T^\nu T^\lambda S^\sigma}$$

(T^ν: tangent vector to geodesic) (S^σ: separation vector)

- <u>Exercise:</u> figure out how big (or small) the BH would have to be, to make an astronaut very <u>un</u>comfy.

▶ Pick $\{S^\mu\}$ of interest, compute Riemann, & plug in numbers, near $r = r_g$ & near $r = 0$.

(e.g. $\frac{D^2 S^t}{d\tau^2} = R^t{}_{rtr} T^r T^t S^r$ + other terms...)

Geodesics for Schwarzschild

Turning the handle for $\Gamma^{\mu}{}_{\lambda\sigma}$ yields

(1.5) $$\frac{d^2t}{d\lambda^2} + \frac{r_g}{r(r-r_g)}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0 \quad ;$$

(1.6) $$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\varphi}{d\lambda}\right)^2 = 0 \quad ;$$

(1.7) $$\frac{d^2\varphi}{d\lambda^2} + \frac{2}{r}\frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} + 2\cot\theta\frac{d\varphi}{d\lambda}\frac{d\theta}{d\lambda} = 0 \quad ;$$

(1.8) $$\frac{d^2r}{d\lambda^2} + \frac{r_g}{2r^3}(r-r_g)\left(\frac{dt}{d\lambda}\right)^2 - \frac{r_g}{2r(r-r_g)}\left(\frac{dr}{d\lambda}\right)^2 - (r-r_g)\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{d\lambda}\right)^2\right] = 0 \quad .$$

Fortunately, the high degree of symmetry allows solving for $\frac{dx^\mu}{d\lambda}$, as follows:-

- Consider φ eqn:

(2.1) $$\left[\frac{d}{d\lambda}\left(r^2\sin^2\theta\,\frac{d\varphi}{d\lambda}\right)\right]\frac{1}{r^2\sin\theta} = \frac{d^2\varphi}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda} + 2\cot\theta\,\frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda} = 0$$

(2.2) $$\Rightarrow \boxed{r^2\sin^2\theta\,\frac{d\varphi}{d\lambda} = L_\varphi} = (\text{const.})$$

- Consider t eqn:

(2.3) $$\left(\frac{d}{d\lambda}\left[\left(1-\frac{r_g}{r}\right)^n\frac{dt}{d\lambda}\right]\right)\frac{1}{(1-r_g/r)^n} = \frac{d^2t}{d\lambda^2} + \frac{n}{(1-r_g/r)}\cdot\frac{r_g}{r^2}\frac{dr}{d\lambda}\frac{dt}{d\lambda}$$

$$\Rightarrow n=1$$

(2.4) so $$\boxed{\left(1-\frac{r_g}{r}\right)\frac{dt}{d\lambda} = E} = (\text{const}'.) =$$

These "first integrals" are available because of SYMMETRY:

(2.5) • Stationary $\Rightarrow$ $\boxed{(K^\mu) = (1,0,0,0)}$ is a K.V.

(2.6) • spherically symmetric $\Rightarrow$ (a) can rotate till particle moves in (x-y) plane i.e. @ $\theta = \pi/2$

(this step works only for 1 particle)

(2.7) (b) $\boxed{R^\mu = (0,0,0,1)}$ is a K.V.

$\uparrow_{\varphi}$ direction

(2.8) In fact: $$\boxed{E = -K_\mu\frac{dx^\mu}{d\lambda}}$$

(2.9) and $$\boxed{L = +R_\mu\frac{dx^\mu}{d\lambda}}$$

Using the $\theta = \pi/2$ argument of (a) above (1-particle) we see that what remains is

(3.1) $$\frac{d^2r}{d\lambda^2} + \frac{r_g}{2r^3}(r-r_g)\left(\frac{dt}{d\lambda}\right)^2 - \frac{r_g}{2r(r-r_g)}\left(\frac{dr}{d\lambda}\right)^2 - (r-r_g)\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{d\lambda}\right)^2\right] = 0$$

and now

$$[\dots] = 0 + \left(\frac{d\varphi}{d\lambda}\right)^2 = \frac{L^2}{r^4}$$

so that

(3.2) $$\frac{d^2r}{d\lambda^2} + \frac{r_g}{2r^2}\left(1-\frac{r_g}{r}\right)^{-1}E^2 - \frac{r_g}{2r^2}\left(1-\frac{r_g}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - \frac{L^2}{r^3}\left(1-\frac{r_g}{r}\right) = 0$$

This actually has a first integral too, most easily computed by realizing that

(3.3) $$\boxed{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = -\epsilon}$$

must be constant along a geodesic (we varied $\sqrt{\ }$ of this to get the geodesic eqn! :-))

For us,

$$-\epsilon = -\left(1-\frac{r_g}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{r_g}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\varphi}{d\lambda}\right)^2$$

$$\epsilon = \left(1-\frac{r_g}{r}\right)^{-1}E^2 - \left(1-\frac{r_g}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - \frac{L^2}{r^2}$$

i.e. $$\left(\frac{dr}{d\lambda}\right)^2 = -\left(\epsilon + \frac{L^2}{r^2}\right)\left(1-\frac{r_g}{r}\right) + E^2 \qquad (\phi)$$

(3.4) $$\Rightarrow \boxed{\frac{dr}{d\lambda} = \mp\sqrt{E^2 - \left(1-\frac{r_g}{r}\right)\left(\epsilon + \frac{L^2}{r^2}\right)}}$$

infalling/outgoing geodesic

▷ Another way to think about this equation is to notice that (ϕ) can be rewritten

(3.5) $$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V_{eff}(r) = \left(\frac{E^2}{2}\right)$$

(like K.E. + P.E. = total energy = conserved)

⇒ $r(\lambda)$ obeys equation of non-relativistic fame where we imagine λ is non-rel time and

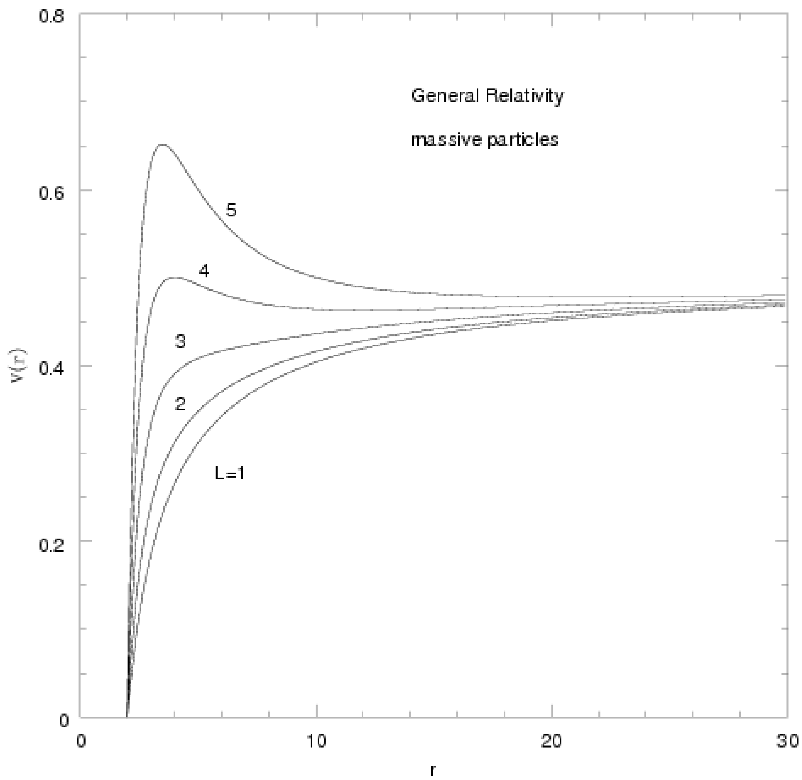
(3.6) $$\boxed{V_{eff} = -\left(1-\frac{r_g}{r}\right)\left(\epsilon + \frac{L^2}{r^2}\right)} = -\epsilon - \frac{L^2}{r^2} + \frac{\epsilon r_g}{r} + \frac{L^2 r_g}{r^3}$$

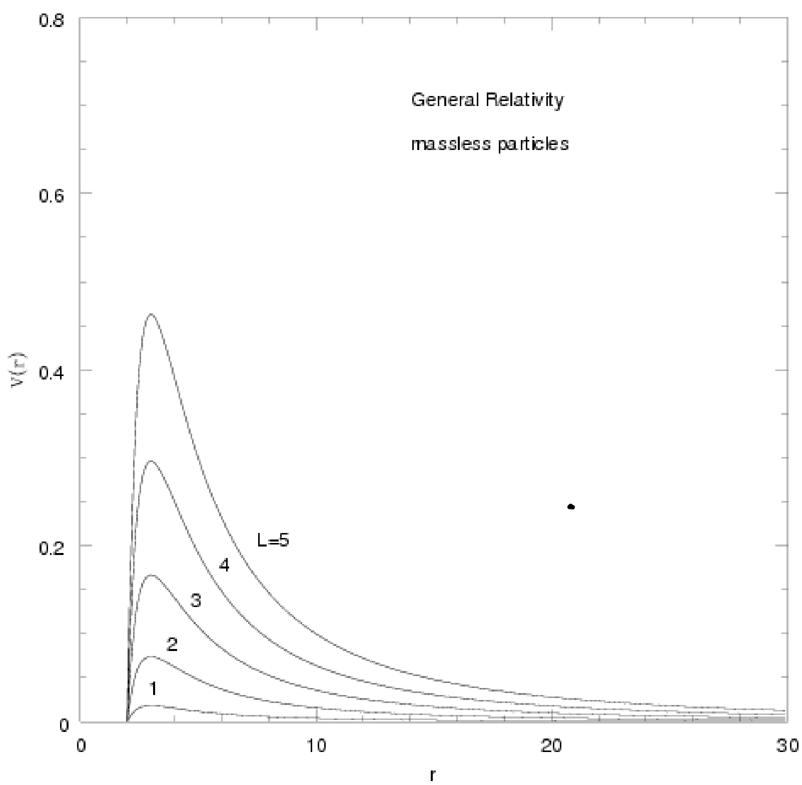
Picturing $V_{eff}(r(\lambda))$

(*) Please Do ask questions; this is crucial stuff! ☺

↓

Discussion of the physics





The question of main interest is whether there are any turning points for the motion; this will tell us whether
- the probe ("test") particle falls in, or
- the probe can do a circular orbit, or
- the probe misses & goes out to $r \to \infty$ again.

Circular orbit

(5.1) For this, we require $\frac{dV_{eff}}{dr} = 0$

(5.2) Now, $\frac{dV_{eff}}{dr} = \frac{2L^2}{r^3} - \frac{\epsilon r_g}{r^2} - \frac{3L^2 r_g}{r^4}$

$= 0$ at $r = r_c$

$\Rightarrow \epsilon r_g r_c^2 - 2L^2 r_c + 3L^2 r_g = 0$

So $r_c = \left(+2L^2 \pm \sqrt{4L^4 - 12L^2 \epsilon r_g^2}\right)\frac{1}{2\epsilon r_g}$

(5.3) $= \frac{L^2}{\epsilon r_g} \pm \frac{L^2}{\epsilon r_g}\sqrt{1 - 3\frac{\epsilon r_g^2}{L^2}}$ when $\epsilon \neq 0$ (*)

and when $\epsilon = 0$ we have

(5.4) $\boxed{r_{c(ICO)} = \frac{3}{2} r_g = 3GM}$ Innermost circular orbit

Expanding (*) for small-ϵ we have

$r_c \simeq \frac{L^2}{\epsilon r_g}\left[1 \pm \left(1 - \frac{3}{2}\frac{\epsilon r_g^2}{L^2}\right) + \mathcal{O}(\epsilon^2)\right]$

(5.6) $= \frac{L^2}{\epsilon r_g}(1 \pm 1) \mp \frac{3}{2} r_g$ $\Rightarrow$ $\ominus$ sqrt sign choice

(5.7) $\Rightarrow$ $\boxed{r_c = \frac{L^2}{\epsilon r_g}\left[1 \mp \sqrt{1 - 3\frac{\epsilon r_g^2}{L^2}}\right]}$ (small-ish L^2) (‡)

larger L^2

For $m^2 > 0$ & $L^2 \gg 1$, there are actually two solutions which lie outside r_{c*}, given by

(5.8) $r_{c,1} \simeq \frac{L^2}{GM}$ and $r_{c,2} \simeq 3GM^{+}$

(6.1) N.B.: $\epsilon = 0$ for massless particles

$\Rightarrow$ photons can orbit forever at $r_c = r_{c*}$ (ONLY!)

Any photon moving a bit in or out must either fall into the black hole or escape to ∞.

It may buzz around the BH (outside of $r = r_g$) a few times before flitting off to $r \to \infty$ ☺

(6.2) (‡) In fact, for $L^2 \gg 1$ this implies that the inner orbit is the unstable one, as it matches with the photonic circular orbit.

$\Rightarrow$ ⊛ the stable orbit is at larger radius! ($L^2 \gg 1$)

- When do the stable & unstable orbits coalesce?

(6.3) When $\sqrt{1 - \frac{3\epsilon r_g^2}{L^2}} = 0$

(N.B.: impossible for photons)

(6.4) $\Rightarrow \quad L^2 = 3\epsilon r_g^2$ so that

(6.5)
$$\boxed{r_{c,\,ISCO} = 3r_g} = 6GM = 2r_{c,\,ICO}$$

(P.S. All of this is analyzed in GR.
If there were an "alternative" theory of gravity, perhaps involving a scalar-tensor story and/or other terms in $\mathcal{L}$, and/or we had different coupling of $g_{\mu\nu}$ to matter fields than we do, then these calculations would need to be redone.)

Precession of Perihelion

- By Birkhoff's theorem the exterior Schwarzschild metric is very important for astrophysical applications. (Great! ☺)
- Orbits in GR actually do <u>not</u> follow conic sections; they are approximately ellipses that precess.
- Computed geodesic equation. Combining $r(\lambda)$ eqn and $\phi(\lambda)$ eqns gives

(1.1) $$\left(\frac{dr}{d\phi}\right)^2 + \frac{r^4}{L^2}\left(1-\frac{r_g}{r}\right) + r^2\left(1-\frac{r_g}{r}\right) = \frac{2\mathcal{E}\,r^4}{L^2} \qquad \left(\mathcal{E} \equiv \tfrac{1}{2}E^2\right)$$

(1.2) Define $x \equiv \dfrac{2L^2}{r_g r}$ (useful abbreviation).

(1.3) Then $$\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{2L}{r_g}\right)^2 - 2x + x^2 - \frac{r_g^2}{2L^2}x^3 = \frac{\mathcal{E}L^2}{2r_g^2}$$

so $$\left(\frac{dx}{d\phi}\right)^2 + \left[\left(\frac{2L}{r_g}\right)^2 - \frac{\mathcal{E}L^2}{2r_g^2}\right] - 2x + x^2 - \frac{r_g^2}{2L^2}x^3 = 0$$

Acting on this with $(d/d\phi)$ gives
$$2\left(\frac{d^2x}{d\phi^2}\right)\left(\frac{dx}{d\phi}\right) - 2\left(\frac{dx}{d\phi}\right) + 2x\left(\frac{dx}{d\phi}\right) - \frac{3r_g}{2L^2}x^2\frac{dx}{d\phi} = 0$$

Cancelling a common factor of $2(dx/d\phi)$ (whew! ☺) gives

(1.4) $$\boxed{\frac{d^2x}{d\phi^2} - 1 + x = \boxed{3\left(\frac{r_g}{2L}\right)^2}x^2}$$ cf. absent for old Newtonian story

- Expand $x = x_0 + x_1$

(1.5a) Newtonian
(1.5b) piece $$\boxed{\frac{d^2x_0}{d\phi^2} = 1 - x_0}$$

$$\boxed{\frac{d^2x_1}{d\phi^2} + x_1 = \underbrace{3\left(\frac{r_g}{2L}\right)^2 x_0^2}_{\text{"source" for } x_1}}$$ Perturbation piece

(1.6a) Solution $\boxed{x_0 = 1 + e\cos\phi}$
ellipse ; $e^2 = 1-(b/a)^2$

(1.6b) Math (Carroll p.215) ⇒
$$\boxed{x_1 = 3\left(\frac{r_g}{2L}\right)^2\left[\underbrace{\left(1+\frac{e^2}{2}\right)}_{\text{constant displacement}} + e\phi\sin\phi - \underbrace{\frac{e^2}{6}\cos 2\phi}_{\text{oscillates about } 0}\right]}$$

Consider $x = 1 + e\phi\cos\phi + \dfrac{3r_g^2}{2L}e\phi\sin\phi$ [x_0 plus 2nd term only in x_1]

$$\cong 1 + e\cos[(1-\alpha)\phi], \quad \alpha = 3(r_g/2L)^2$$

(1.7) where $$\boxed{\Delta\phi = 2\pi\alpha = \frac{6\pi G^2M^2}{L^2c^4}}$$ cf. famed <u>precession of perihelion of Mercury</u>: 43" per century

(1.8) and ex x_0 $$\boxed{L^2 \cong GM(1-e^2)a}$$