Quick review and Today's Plan

2005-11-23 [+tgs!] ⓪

Note Title 19/11/2005

- **Last time:** We set up solving Einstein's equations for a
(0.1) perfect fluid, which has $\boxed{T_{\mu\nu} \doteq (\rho+p)\, u_\mu u_\nu + p\, g_{\mu\nu}}$.
- When system is static, and spherically symmetric, pick
(0.2) metric Ansatz: $\boxed{ds^2 = -e^{2\alpha}dt^2 + e^{2\beta}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)}$
(0.3) - $\boxed{\nabla_\mu T^{\mu\nu} = 0}$ relates dp/dr to $d\rho/dr$ and $d\alpha/dr$, so that
(0.4) the system $\{\alpha(r), \beta(r), \rho(r), p(r)\}$ is solved via $G_{\mu\nu} = \frac{8\pi G_N}{c^4} T_{\mu\nu}$.

$\Rightarrow$ TOV equations for hydrostatic equilibrium ☺

⊛ Today's material enables matching TOV star to BH outside!

- **Today:** the Schwarzschild black hole, which is a solution of the vacuum Einstein equations $\boxed{T_{\mu\nu} \equiv 0}$.
  - This is feasible because classical GR is nonlinear theory (unlike EM).

  - Important places in this spacetime:-
(1) horizon, (2) singularity;
(3) $\boxed{\text{geodesics} \to \text{next time.}}$

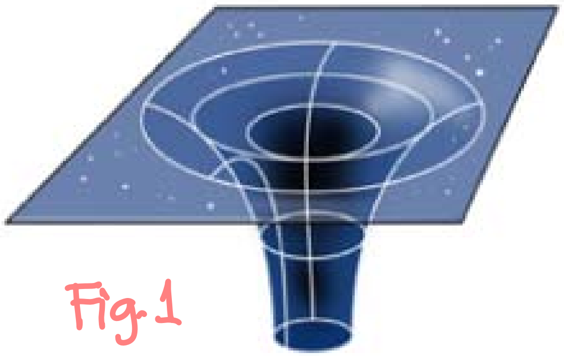


Fig. 1

Really big stars have deaths ending in black holes ⤴.

Newtonian intuition

(1.1) • Try to escape from gravitational pull $\boxed{\Phi_N = -\frac{Gm}{r}}$ $(r \geq R)$

Non-relativistic kinetic energy

(1.2) of liftoff → grav. P.E. i.e. $\boxed{\frac{1}{2}mv_{esc}^2 = +\frac{Gm}{r}}$

(1.3) • Escape velocity → c as $r \to$ $\boxed{r_g = \frac{2Gm}{c^2}}$

This is our figure of merit: roughly telling us the coordinate radius at which GR nonlinearity will play a strong rôle. We will soon see this confirmed →

Quantum gravity intuition

• An electron is not called a black hole. Why not ?

⊛ Basic criterion for being a black hole: $\boxed{\lambda_c < r_g}$

Compton wavelength of object (e.g. electron)

(1.4) $\boxed{\lambda_c = \frac{h}{mc}}$, while $\boxed{r_g = \frac{2Gm}{c^2}}$.

For e^- : $\lambda_c \sim 10^{-12}\,m$
$r_g \sim 10^{-57}\,m$

Rearranging ⇒ $\frac{r_g}{\lambda_c} = \frac{2Gm^2}{hc}$ (dimensionless ☺ ✓)

(1.5) • Crossover mass: (where $\lambda_c = r_g$) $\boxed{m_{Pl} = \sqrt{\frac{hc}{2G}}}$ Planck mass. $\sim 10^{-5}\,g$ (small virus size)

Anatomy of a Black Hole

- Schwarzschild is one of the most important solutions of GR, because it has been useful both experimentally and theoretically. (e.g. of latter: explosion in finding new 'BH's in last ~ decade, in string theory context).

- In the most well-known coordinate system, its metric is

(2.1) $$\boxed{ds^2 = -(1-r_g/r)\,dt^2 + \frac{dr^2}{(1-r_g/r)} + r^2(d\theta^2+\sin^2\theta\, d\varphi^2)}$$

where

(2.2) $$\boxed{r_g \equiv \frac{2Gm}{c^2}}$$ ← "Schwarzschild radius"

- Far away from the black hole, we see a Newtonian

(2.3) potential $\Phi_N \simeq -\frac{Gm}{r}$, and spacetime becomes flat as $r \to \infty$.

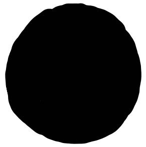
- The locus $r=0$ is a place where all metric components seem to go haywire: sizes of $\perp$ S^2 are zero, and

(2.4) $(1-r_g/r) \to -\infty$ as $r \to 0$.

?! @?

$\Rightarrow$ Let's analyse $r=0$ & r_g!

Event Horizon



- Schwarzschild coords break down as $r \to r_g$, because $g_{tt} \to 0$ and $g_{rr} \to \infty$ there. This turns out to be a problem with the coordinate system, rather than with the BH, actually, as we will see in lecture after next.

(3.1) • Terminology: <u>event horizon</u> where $\boxed{g^{rr} \to 0}$ *

(3.2) • For Schwarzschild, this is at $\boxed{r = r_g}$
and at $r = r_g$ the area of our
(3.3) transverse S^2 ((θ, φ) coords) is finite: $\boxed{A_h = 4\pi r_g^2 = \frac{16\pi G^2 m^2}{c^4}}$

e.g. solar mass $m_\odot \sim 10^{30}$ kg $\Rightarrow$ $r_g \sim$ few km
$\Rightarrow$ horizon size for macroscopic BH <u>is</u> macroscopic.
e.g. Earth-mass $\Rightarrow$ $r_g \sim$ ping-pong ball size.

- <u>QUESTION</u>: are we inside a giant BH right now?
Quite possibly - bigger black holes are less "dense" (!)
(3.4) (rough back-of-envelope estimate: $\rho \sim \frac{\text{Mass}}{\text{Vol}} \sim \frac{M}{4/3\,\pi r_g^3} \propto \frac{1}{m^2}$)

(* <u>NOT</u> $g_{tt} \to 0$, which is commonly & mistakenly believed...)

Curvature Singularity

- Some drama queens insist that Einstein's theory was "doomed" from the beginning, because it "contained the seeds of its own destruction"! I prefer to say that
  ⊛ Einstein's theory of gravity, our friend GR, works well as an effective theory — ONLY when all curvature invariants are small, in Planck units.

(4.1) Require $$\boxed{|R|\,\ell_P^2 \ll 1, \quad |R^{\mu\nu}R_{\mu\nu}|\,\ell_P^4 \ll 1, \quad |R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}|\,\ell_P^4 \ll 1, \ \ldots}$$

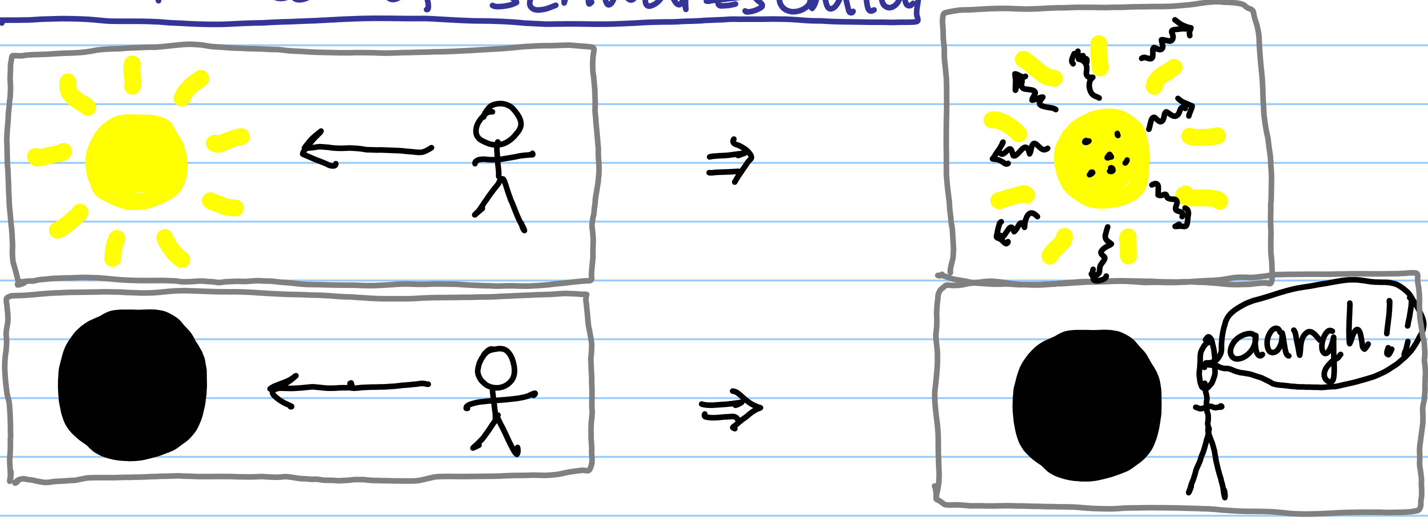
← grtensor v. useful for this

These are statements we can all agree upon, because they are all dimensionless, scalar statements. ☺

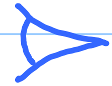
(4.2) • Singularity of Schwarzschild is at the core: $\boxed{r=0}$
(4.3) where, for example, $$\boxed{R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} = -\frac{12 r_g^2}{r^6}}$$
(4.4) while $\underbrace{R \equiv 0 \text{ and } R_{\mu\nu} \equiv 0}$ (!)

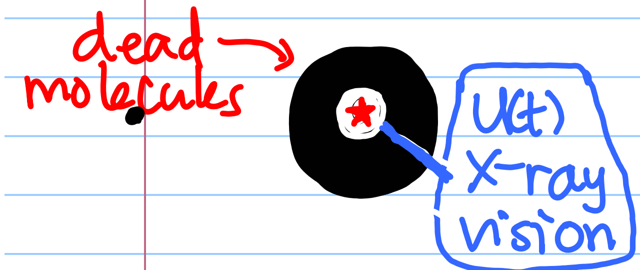
[This follows from $T_{\mu\nu} \Rightarrow G_{\mu\nu}=0 \Rightarrow R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R$
so, tracing this gives $R = \frac{1}{2}(D)R$ ie. $R=0$ (unless $D=2$)
Substituting $R=0$ gives $R_{\mu\nu} = \frac{1}{2} g_{\mu\nu}(\emptyset) \equiv 0$. ■]

Tidal Forces of Schwarzschild



(whoops! ☺)





- Astronauts falling in to stars get fried;
- those going into black holes first get hung, drawn and quartered by <u>tidal forces</u>.

- Recall separation vector S^μ of neighbour geodesics:

$$A^\mu = \frac{D}{d\tau} V^\mu = \boxed{\frac{D^2 S^\mu}{d\tau^2} = R^\mu{}_{\nu\lambda\sigma} T^\nu T^\lambda S^\sigma}$$

(separation vector)

(tangent vector to geodesic)

- <u>Exercise</u>: figure out how big (or small) the BH would have to be, to make an astronaut very <u>un</u>comfy.

▶ Pick $\{S^\mu\}$ of interest, compute Riemann, & plug in numbers, near $r = r_s$ & near $r = 0$.

(e.g. $\frac{D^2 S^t}{d\tau^2} = R^t{}_{rtr} T^r T^t S^r$ + other terms...)

- Maple says: ✓ $R_{trtr} = \frac{-r_g}{r^3}$ $\quad R_{t\theta t\theta} = \frac{r_g(r-r_g)}{2r^2}$ ✓

✓ $R_{t\varphi t\varphi} = \frac{r_g(r-r_g)\sin^2\theta}{2r^2}$ $\quad R_{r\theta r\theta} = \frac{-r_g}{2(r-r_g)}$ ✓

✓ $R_{r\varphi r\varphi} = \frac{-r_g \sin^2\theta}{2(r-r_g)}$ $\quad R_{\theta\varphi\theta\varphi} = r\sin^2\theta \, r_g$

- Potentially "dangerous" bits around the horizon ⇒ let's analyze.

⊛ $$\frac{D^2S^\mu}{d\tau^2} = R^\mu{}_{\nu\lambda\sigma} U^\nu U^\lambda S^\sigma$$

Now, $R^r{}_{\varphi r\varphi} = g^{rr} R_{r\varphi r\varphi}$ ∵ metric diagonal

$$= (1-r_g/r) \cdot \frac{-r_g\sin^2\theta}{2(r-r_g)} = \frac{-1}{2}\frac{r_g}{r}\sin^2\theta$$

while $R^r{}_{\theta r\theta} = \frac{-1}{2}\frac{r_g}{r}$

and $R^t{}_{\varphi t\varphi} = \frac{-1}{(1-r_g/r)} \cdot \frac{r_g}{2r}\frac{(r-r_g)\sin^2\theta}{r} = \frac{-1}{2}\frac{r_g}{r}\sin^2\theta$ (!)

whereas $R^t{}_{rtr} = \frac{-1}{(1-r_g/r)} \cdot \frac{-r_g}{r^3} = +\frac{r_g}{r^2(r-r_g)}$ (?)

and $R^t{}_{\theta t\theta} = \frac{-1}{2}\frac{r_g}{r}$ while finally $R^\theta{}_{\varphi\theta\varphi} = \frac{r_g}{r}$.

- So danger lurks only in $R^t{}_{rtr}$ near horizon.