## $\boxed{\text{Birkhoff's Theorem}}$

In $D=4$, possible to prove via Einstein's equations that

- if spherical symmetry holds

and

- if outside matter distribution

then

▷ metric outside is Schwarzschild.

Nontrivial uniqueness theorem, specific to $d=4$. Rather involved proof — see §5.2 of Carroll for full story.
Some <u>physical</u> details are worth picking out, as follows :—

▷ Every spherically symmetric vacuum metric has a Killing Vector that is timelike

In general, a spacetime with a timelike K.V. — near $\infty$ — is called <u>stationary</u>.

["doing exactly the same thing @ every time"]

c.f. <u>static</u> spacetime

["not doing anything"]

Let's check this by looking at the K.V. equation

(5.1) $$\nabla_{(\mu} V_{\nu)} = 0$$

For a spherically symmetric metric we can write

$$ds^2 = -e^{2\alpha} dt^2 + e^{2\beta} dr^2 + r^2 d\Omega_2^2$$

Putting this in Einstein's equations shows that

$\alpha = \alpha(r)$ and $\beta = \beta(r)$ (no $t$ dependence)

We need $\Gamma^{\mu}{}_{\lambda\sigma}$ in order to write the K.V. equations.

(5.2)
$$\begin{cases} \Gamma^t{}_{tr} = (\partial_r \alpha) \;; & \Gamma^r{}_{tt} = e^{2(\alpha-\beta)}(\partial_r \alpha) \;; \\ \Gamma^r{}_{rr} = (\partial_r \beta) \;; & \Gamma^\theta{}_{r\theta} = \frac{1}{r} = \Gamma^\varphi{}_{r\varphi} \;; \\ \Gamma^r{}_{\theta\theta} = -r e^{-2\beta} \;; & \Gamma^r{}_{\varphi\varphi} = -r\sin^2\theta\, e^{-2\beta} \;; \\ \Gamma^\theta{}_{\varphi\varphi} = -\sin\theta\cos\theta \;; & \Gamma^\varphi{}_{\theta\varphi} = \cot\theta \;. \end{cases}$$

(5.3)
$$\nabla_{(\mu} V_{\nu)} = \partial_{(\mu} V_{\nu)} - \Gamma^\lambda{}_{\mu\nu} V_\lambda \qquad \left(\leftarrow \tfrac{1}{2}(4)(5) = 10 \text{ indep. cpts}\right)$$

$$\begin{array}{l|l} \nabla_{(t} V_{t)} = \cancelto{0}{\partial_t V_t} - e^{2(\alpha-\beta)}(\partial_r \alpha) V_r \;; & \nabla_{(r} V_{\theta)} = \frac{1}{2}\partial_r V_\theta - \frac{1}{r} V_\theta \;; \\ \nabla_{(t} V_{r)} = \frac{1}{2}\partial_r V_t - (\partial_r \alpha) V_t \;; & \nabla_{(r} V_{\varphi)} = \frac{1}{2}\partial_r V_\varphi - \frac{1}{r} V_\varphi \;; \\ \nabla_{(t} V_{\varphi)} = 0 \;;\quad \nabla_{(t} V_{\theta)} = 0 \;; & \nabla_{(\theta} V_{\theta)} = -r e^{-2\beta} V_r \;; \\ \nabla_{(r} V_{r)} = \partial_r V_r - (\partial_r \beta) V_r \;; & \nabla_{(\theta} V_{\varphi)} = -\cot\theta\, V_\varphi \;; \end{array}$$

and (last but not least)

$$\nabla_\varphi V_\varphi = + \sin\theta\cos\theta\, V_\theta + r\sin^2\theta\, e^{-2\beta} V_r .$$

⑥

(6.1) ① $\Rightarrow \nabla_t V_t = 0$, $V_r = 0$. $\Rightarrow$ ⑤, ⑧ work.

⑩ Then $\Rightarrow V_\theta = 0$. $\Rightarrow$ ⑥ works.

② $\Rightarrow$ $\dfrac{\partial_r V_t}{V_t} = 2\partial_r \alpha$ and since $\partial_\mu = \delta^r_\mu \partial_r$

($\because$ spherically symmetric & obeys vacuum Einstein)

so $\ln V_t = 2\ln\alpha + c$.

(6.2) $V_t = c_1 \alpha^2(r)$

(6.3) ⑨ $\Rightarrow V_\varphi = 0 \Rightarrow$ ⑦ works

⑩ also works.

(6.4) $\Rightarrow \boxed{(V^\mu) = (c_1\alpha^2(r), 0, 0, 0)}$

This is timelike (for $r > 2GM$, to be precise).

## Four K.V.

There are actually 3 more K.V. for Schwarzschild, corresponding to 3 rotational symmetry generators.

$\Rightarrow$ total of ④ : $\left\{ \begin{array}{l} 1 \text{ timelike} \\ 3 \text{ spacelike} \end{array} \right\}$ (outside $r_g$)

$c^3$

* What kind of matter is ok, and what's the equation of state (?)

- Let's suppose we have a perfect fluid

(2.2) $$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p\, g_{\mu\nu}$$

and try to solve Einstein equations in the star interior. [At minimum, we know that at $r = R$ the metric should match on to exterior-Schwarzschild]

Make Ansätz

(2.3) $$\boxed{ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega_2^2}$$

Choice of radial coörd such that radius $(S^2) = r$

Looking for solutions of

(2.4) $$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = T_{\mu\nu} \cdot 8\pi G$$

Need to know Einstein tensor for spacetime (2.3).

We can either turn the handle by hand, or use Maple or a trusted source for general sph. sym. (2-3)'s.

Find

(3.1)
$$\begin{cases} G_{tt} = \frac{1}{r^2} e^{2(\alpha-\beta)} \left( 2r\partial_r\beta - 1 + e^{2\beta} \right) \\ G_{rr} = \frac{1}{r^2} \left( 2r\partial_r\alpha + 1 - e^{2\beta} \right) \\ G_{\theta\theta} = r^2 e^{-2\beta} \left[ \partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{1}{r}\partial_r(\alpha-\beta) \right] \\ G_{\varphi\varphi} = \sin^2\theta \, G_{\theta\theta} \end{cases}$$

For a timelike fluid in its rest frame

(3.2) $$U_\mu = (e^\alpha, 0, 0, 0)$$ is unit-normalized and timelike.

With (3.2), find

(3.3)
$$\begin{cases} T_{tt} = (e^{2\alpha})\rho \\ T_{rr} = (e^{2\beta})\, p \\ T_{\theta\theta} = (r^2)p \\ T_{\varphi\varphi} = (r^2\sin^2\theta)p \end{cases}$$

(3.4) Have $\underline{tt}$ $$\frac{1}{r^2} e^{-2\beta}\left[2r\partial_r\beta - 1 + e^{2\beta}\right] = 8\pi G \rho(r)$$

(3.5) $\underline{rr}$ $$\frac{1}{r^2} e^{-2\beta}\left[2r\partial_r\alpha + 1 - e^{2\beta}\right] = 8\pi G p(r)$$

(3.6) $\underline{\theta\theta}$ $$e^{-2\beta}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\,\partial_r\beta + \frac{1}{r}\partial_r(\alpha-\beta)\right] = 8\pi G p(r)$$

( $\underline{\varphi\varphi}$ proportional to $\theta\theta$ eqn $\because$ spherical symmetry)

Turns out: easiest way forwards is to define
$$m(r) \equiv \frac{r}{2G}\left(1 - e^{-2\beta}\right)$$

(3.7) or $$\boxed{e^{2\beta} \equiv \left(1 - \frac{2Gm(r)}{r}\right)^{-1}}$$

The $tt$ equation is then

(4.1) $$\boxed{\frac{dm(r)}{dr} = 4\pi r^2 \rho(r)}$$ (sensible! ✓)

Now: if we want to integrate up to get the mass, there is a wee subtlety. Namely, that to match the interior and exterior metrics what we should do is

(4.2) write $$\bar{M} \equiv \int_0^R 4\pi (r')^2 \, d(r') \, \rho(r') \left(1 - 2\frac{Gm(r)}{r}\right)^{-1/2}$$

because this is going to match $M_{ADM}$ for Schwarzschild. The function

(4.3) $$m(r) \equiv (4\pi) \int_0^r dr' \, (r')^2 \rho(r') \quad ; \quad M \equiv m(R)$$

is the "naive" mass ; then

(4.4) $$\bar{M} - M = E_{binding}$$ ← notion that is well-defined for these stars, but not always in general in GR.

The other equation was for $\alpha$

(4.5) $$\frac{d\alpha}{dr} = \left(\frac{Gm(r) + 4\pi G r^3 p(r)}{r^2 [1 - 2Gm(r)/r]}\right)$$

Can be integrated → $g_{tt} = -e^{2\alpha}$

Also, can find $dp/dr$ via:

(4.6) $$\nabla_\mu T^{\mu\nu} = 0$$

to an equation which is simpler to work with, via

(4.7) $$\boxed{(\rho + p) \frac{d\alpha}{dr} = -\frac{dp}{dr}} ,$$

viz :-

(4.8) $$\boxed{\frac{dp}{dr} = -\frac{(\rho(r) + p(r))\,[Gm(r) + 4\pi G r^3 p(r)]}{r\,[r - 2Gm(r)]}}$$

Eqns (4.1) & (4.8) are together called the Tolman-Oppenheimer-Volkoff equation

or, less fancily, the eqns for hydrostatic equilibrium when Einstein gravity is your spacetime theory. ☺

(4.9) To solve, we need $\boxed{p(\rho)}$ : "equation of state"

Astrophysically interesting systems often involve fluids with polytropic equations of state,

(5.1) $p = K\rho^\gamma$ (e.g. "$p = w\rho$" [simplest]) constant

See Carroll on p.234 for working-out of somewhat unrealistic case of incompressible fluid.

(5.2) Take $\rho(r) = \begin{cases} \rho_*, & r \le R \\ 0, & r > R \end{cases}$

Then easy to integrate for $m(r)$:

(5.3) $m(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_*, & r \le R \\ \frac{4}{3}\pi R^3 \rho_*, & r > R \end{cases}$

Hydrostatic equilibrium eqn integrates immediately to give

(5.4) $$\boxed{\frac{p(r)}{\rho_*} = -\frac{\left(\sqrt{R^3 - r_g r^2} - R\sqrt{R - r_g}\right)}{\left(\sqrt{R^3 - r_g r^2} - 3R\sqrt{R - r_g}\right)}} \qquad r \le R$$

where

(5.5) $$r_g \equiv 2GM = \frac{8}{3}\pi R^3 \rho_* G$$

(Notice that the pressure goes up monotonically with decreasing radius; this makes sense because the stuff nearer the core has the weight of the stuff above it to cope with.)

$$p(0) \to \infty \quad \text{iff} \quad M > \boxed{M_{max} = \frac{4R}{9G}}$$

This is called Buchdahl's theorem.