GR 29 Oct

Note Title 27/10/2004

Raychaudhuri's equation

Our Ricci tensor for spacetime is determined by $T_{\mu\nu}$ via Einstein's equations. What about the effect on matter of spacetime curvature?

(1.1a) (1.1b) Consider a $\underbrace{\text{congruence}}$ of geodesics $\begin{cases} u^\mu u_\mu = -1 \\ u^\lambda \nabla_\lambda u^\mu = 0 \end{cases}$

Set of curves in an open region of spacetime, such that every point in the region lies on precisely one curve.

Previously, we found that for the separation vector

(1.2) $$\frac{DV^\mu}{d\tau} = u^\nu \nabla_\nu V^\mu = B^\mu{}_\nu V^\nu$$

$$\boxed{B^\mu{}_{\cdot\nu} = \nabla_\nu u^\mu}$$

Consider space of vectors $\perp$ and $\parallel$ to u^μ.

- Use projector

(1.3) $$\boxed{P^\mu{}_\nu = \delta^\mu{}_\nu + u^\mu u_\nu}$$

to get the component of a tensor $\perp$ to u^μ.

- Claim: $B_{\mu\nu}$ is in the $\perp$ subspace.

Proof: $u^\mu B_{\mu\nu} = u^\mu \nabla_\nu u_\mu = 0 \because u^\mu u_\mu = -1$ so $u^\mu \nabla_\nu u_\mu = 0$

and $u^\nu B_{\mu\nu} = u^\nu \nabla_\nu u_\mu = 0$ by geodesic equation.

- $B_{\mu\nu}$ has trace, symmetric traceless, and antisymmetric components.

$\boxed{\text{Note}}$: since $B \perp u$, use $P^\mu{}_\nu$ to take the trace! (cute mathematical trick / observation)

(1.4) Define $\theta \equiv P^{\mu\nu} B_{\mu\nu} = \boxed{\nabla_\mu u^\mu = \theta} \leftarrow$ "expansion" of congruence

Then define

$$\sigma_{\mu\nu} = B_{(\mu\nu)} - P_{\mu\nu} \cdot \frac{\theta}{3}$$

there are 3 independent basis vectors in the $\perp$ subspace (not 4 !)

i.e.

(2.1) $$\boxed{\sigma_{\mu\nu} = \nabla_{(\nu} U_{\mu)} - (\delta^\mu{}_\nu + U^\mu U_\nu)\frac{1}{3}(\nabla_\lambda U^\lambda)}$$

"shear" of the congruence

and thirdly define

(2.2) $$\boxed{\omega_{\mu\nu} = \nabla_{[\nu} U_{\mu]}}$$ "rotation" of the congruence

For evolution along the path,

$$\frac{D}{d\tau} = U^\sigma \nabla_\sigma$$

Acting on $B_{\mu\nu}$ we have

$$\frac{D}{d\tau} B_{\mu\nu} = U^\sigma \nabla_\sigma \nabla_\nu U_\mu$$

$$= U^\sigma (\nabla_\nu \nabla_\sigma U_\mu + R^\lambda{}_{\mu\nu\sigma} U_\lambda)$$

$$= \nabla_\nu (U^\sigma \nabla_\sigma U_\mu) - (\nabla_\nu U^\sigma)(\nabla_\sigma U_\mu) - R_{\mu\lambda\nu\sigma} U^\sigma U^\lambda$$

(2.3) $$= -B^\sigma{}_\nu B_{\mu\sigma} - R_{\mu\lambda\nu\sigma} U^\sigma U^\lambda$$

Tracing over this gives

(2.4) $$\boxed{\frac{D\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}U^\mu U^\nu}$$

This is called "RAYCHAUDHURI'S EQUATION"

Other parts (for $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$) are in Appendix F of Carroll on p.461. (They don't get used nearly as often as (5.3).)

▷ How about physical consequences?

- Since $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ are both spatial tensors, (because $B_{\mu\nu} = \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{3} P_{\mu\nu}\theta \perp u^\mu \leftarrow$ timelike)

(3.1) $$\omega_{\mu\nu}\omega^{\mu\nu} \geq 0 \quad \text{and} \quad \sigma_{\mu\nu}\sigma^{\mu\nu} \geq 0 .$$

- If u^μ is orthogonal to a family of hypersurfaces

(3.2) then ($u_{[\mu}\nabla_\nu u_{\rho]} = 0$ and) $\omega_{\mu\nu} \equiv 0$.

- Let's also consider

$$R_{\mu\nu} u^\mu u^\nu$$

Since $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu} - \Lambda g_{\mu\nu}$,

(3.3) $$\therefore R\left(1 - \frac{D}{2}\right) = 8\pi G_N T - \Lambda D ,$$

$(D>2)$

$$R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \frac{2}{(2-D)} \left(8\pi G_N T - \Lambda D\right) + 8\pi G_N T_{\mu\nu} - \Lambda g_{\mu\nu}$$

$$= \frac{1}{(2-D)} g_{\mu\nu} 8\pi G_N T - \frac{\Lambda D g_{\mu\nu}}{(2-D)} + 8\pi G_N T_{\mu\nu} - \Lambda g_{\mu\nu}$$

(3.4) $$= 8\pi G_N \left(T_{\mu\nu} - \frac{T g_{\mu\nu}}{(D-2)}\right) + g_{\mu\nu}\Lambda \frac{2}{(D-2)}$$

(3.5) $$\Rightarrow R_{\mu\nu} u^\mu u^\nu = 8\pi G_N \left[T_{\mu\nu} u^\mu u^\nu + \frac{T}{(D-2)}\right] - \frac{2}{(D-2)}\Lambda$$

This implies that

(3.6) if
- $\omega_{\mu\nu} = 0$
- $T_{\mu\nu} u^\mu u^\nu \geq -\frac{T}{(D-2)} u^\mu u_\nu \leftarrow$ (Strong Energy Condition)
- $\Lambda \leq 0$

then $\frac{d\theta}{d\tau} \leq 0$

and the geodesics converge!

If $\Lambda > 0$ this helps push geodesics apart, which happens owing to (accelerated) expansion.

ENERGY CONDITIONS

(4.1) For a perfect fluid, $T^{(p.f.)}_{\mu\nu} = (\rho+p)u_\mu u_\nu + p g_{\mu\nu}$

Pressure is isotropic so $T_{\mu\nu}t^\mu t^\nu \geq 0$ $\forall$ timelike t^μ
if both $T_{\mu\nu}u^\mu u^\nu \geq 0$
and $T_{\mu\nu}\ell^\mu \ell^\nu \geq 0$ $\exists$ null vector ℓ^μ.

Regard
$$\begin{cases} T_{\mu\nu}u^\mu u^\nu = \rho \\ T_{\mu\nu}\ell^\mu\ell^\nu = (p+\rho)\,(u_\mu \ell^\mu)^2 \end{cases}$$

so we're just said $T_{\mu\nu}t^\mu t^\nu \geq 0 \Rightarrow \rho \geq 0$ & $(p+\rho) \geq 0$.

This is called the Weak Energy Condition.
Others include (& perfect fluid examples :-) :-

(4.2) • WEC : weak energy condition
$T_{\mu\nu}t^\mu t^\nu \geq 0$, $\forall t^\mu$ timelike vectors
($\rho \geq 0$ and $p+\rho \geq 0$)

(4.3) • NEC: null energy condition
$T_{\mu\nu}\ell^\mu\ell^\nu \geq 0$ $\forall \ell^\mu$ null vectors
($p+\rho \geq 0$; note that here ρ can be <0)

(4.4) • DEC: dominant energy condition
$T_{\mu\nu}t^\mu t^\nu \geq 0$ $\forall t^\mu$ timelike AND $T_{\mu\nu}T^\nu{}_\lambda t^\mu t^\lambda \geq 0$
i.e. $T^{\mu\nu}t_\mu$ non-spacelike. ($\rho \geq |p|$).

(4.5) • NDEC: null dominant energy condition
$T_{\mu\nu}\ell^\mu\ell^\nu \geq 0$ $\forall \ell^\mu$ null vector AND $T^{\mu\nu}\ell_\mu$ non-spacelike.
Same as DEC but allows $p=-\rho$.

(4.6) • SEC : strong energy condition
$T_{\mu\nu}t^\mu t^\nu \geq [(D-2)/2]\, T^\lambda{}_\lambda t^\sigma t_\sigma$ $\forall t^\mu$ timelike
(gravity attractive!) ($p+\rho \geq 0$ and $3p+\rho \geq 0$)