## Recap: our action for $g_{\mu\nu}$ field

→ Last time, we started on deriving Einstein's equations for GR from an action principle.

★ Experiment ⇒ massless, spin-two field $\boxed{g_{\mu\nu}}$ (rather than some other type of field.)

★ Lowest-order-in-derivatives action consistent with symmetries (and Occam's Razor) ⇒

(1.1) $$\boxed{S_{grav} = \frac{1}{16\pi G_N}\int d^Dx \sqrt{-g}\,(R - 2\Lambda)}$$

• We began to find eqns of motion by first focusing on

(1.2) $$\boxed{\delta\sqrt{-g} = -\tfrac{1}{2}\sqrt{-g}\, g_{\mu\nu}\,\delta g^{\mu\nu} \quad \left(= +\tfrac{1}{2}\sqrt{-g}\, g^{\mu\nu}\,\delta g_{\mu\nu}\right)}$$

• If we can show this, then this follows, because

(1.3) $$\delta(g^{\mu\nu} g_{\nu\lambda}) = (\delta g^{\mu\nu})\, g_{\nu\lambda} + g^{\mu\nu}(\delta g_{\nu\lambda})$$

(1.4) $$= \delta(g^{\mu}{}_{\lambda}) = \delta(\delta^{\mu}{}_{\nu}) \equiv 0$$

(1.5) $$\Rightarrow \boxed{(\delta g^{\mu\nu})\, g_{\nu\lambda} = -\, g^{\mu\nu}(\delta g_{\nu\lambda})}$$

} "converts upstairs to downstairs" ☺

## Varying metric determinant w.r.t. metric tensor

So what of $\delta\sqrt{-g}$?

- If we consider a matrix $M = (g_{\alpha\beta})$ and take the determinant:

(2.1) $$\det M = \prod_i \lambda_i \quad , \{\lambda_i\} = \text{eigenvalues of } M$$

$$= \exp\left(\ln \prod_i \lambda_i\right)$$

(2.2) $$= \exp\left(\sum_i \ln \lambda_i\right)$$

$$= \exp(\mathrm{Tr}(\ln M)) \quad \leftarrow \left[\text{definition of } \ln M\right]$$

- Now let's vary w.r.t. $M$. We have that

(2.3) $$\delta(\det M) = (\det M) \cdot \delta(\mathrm{Tr}(\ln M)) \quad \left[\delta(\ln X) = \frac{\delta X}{X}\right]$$

i.e.

(2.4) $$\frac{\delta(\det M)}{(\det M)} = \mathrm{Tr}(M^{-1}\delta M) \quad \leftarrow \left[\text{cyclicity of Tr}\right]$$

- And here we take $\det M = (-g)$ so

(2.5) $$\frac{\delta(-g)}{(-g)} = g^{\mu\nu}\delta g_{\mu\nu} = -g_{\alpha\beta}\,\delta g^{\alpha\beta}$$

(2.6) so that $$\boxed{\frac{\delta\sqrt{-g}}{\sqrt{-g}} = \frac{-1}{2} g_{\alpha\beta}\,\delta g^{\alpha\beta}} \quad \left(= +\frac{1}{2} g^{\alpha\beta}\,\delta g_{\alpha\beta}\right)$$

## Varying Ricci scalar w.r.t. metric tensor

- The trickiest part is finding $\delta R$, and we can simplify that by writing

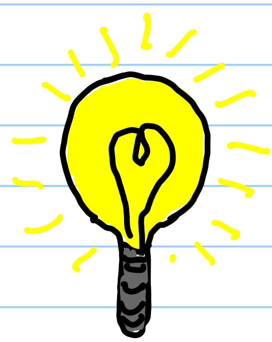
$$\delta(R) = \delta(g^{\mu\nu} R_{\mu\nu})$$

(3.1)
$$= (\delta g^{\mu\nu}) R_{\mu\nu} + g^{\mu\nu} (\delta R_{\mu\nu})$$

[Leibniz rule]

this we know

this we want



(*) Easier (MUCH!) to find $\frac{\delta R_{\mu\nu}}{\delta \Gamma^{\alpha}{}_{\beta\gamma}}$ than to find $\frac{\delta R_{\mu\nu}}{\delta g_{\alpha\beta}}$ largely because $\Gamma^{\cdot}_{\cdot\cdot}$ is first-order in derivatives of $g_{\cdot\cdot}$ while $R_{\cdot\cdot}$ is second-order.

- The only potential fly in the ointment here is: what if the fantasy that $\delta\Gamma^{\cdot}_{\cdot\cdot}$ is independent of the $\delta g_{\cdot\cdot}$ is actually just that: fantasy ??

## How varying Christoffels helps

- The super-cool observation of Palatini was that insisting on a torsion-free, metric-compatible connection does not bugger up anything ($\nabla$). ☺
- Some authors of GR texts call this procedure "letting $\delta\Gamma$ flap in the breeze" and I like this terminology. More accurately, it's called using the "first-order formalism"; insisting $\nabla_\alpha g_{\beta\gamma} = 0$ after the fact just neatly connects $\Gamma^\alpha{}_{\beta\gamma}$ to $g_{\mu\nu}$ (in the way we know).
- OK. So let's go compute $\dfrac{\delta R^\alpha{}_{\beta\gamma\delta}}{\delta \Gamma^\mu{}_{\nu\lambda}}$ !
- Possible worry: $\Gamma^\mu{}_{\nu\lambda}$ is NOT a tensor (as we saw in lecture previously, there are terms in the transformation law for $\Gamma^\mu{}_{\nu\lambda}$ which spoil tensoriality. ..-).
- Resolution: interestingly, these "nasty bits" actually go away if you take a difference of two $\Gamma$'s !

In particular:

(4.1) $\boxed{\delta\Gamma^\mu{}_{\nu\lambda} \text{ is a tensor.}}$ (Try it & see!)

## Final steps toward $\delta R$

(5.1) • We're interested in $R^{\alpha}{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^{\alpha}{}_{\delta\beta} + \Gamma^{\alpha}{}_{\gamma\epsilon}\Gamma^{\epsilon}{}_{\delta\beta} - (\gamma \leftrightarrow \delta)$

(5.2) • By taking $\nabla_\gamma(\Gamma^{\alpha}{}_{\delta\beta}) = \left\{\partial_\gamma(\Gamma^{\alpha}{}_{\delta\beta}) + \Gamma^{\alpha}{}_{\gamma\sigma}(\delta\Gamma^{\sigma}{}_{\delta\beta}) - \Gamma^{\sigma}{}_{\gamma\delta}(\delta\Gamma^{\alpha}{}_{\sigma\beta})\right\} - \left\{(\gamma \leftrightarrow \delta)\right\}$

it is straightforward* to find that

(5.3) $$\boxed{\delta R^{\alpha}{}_{\beta\gamma\delta} = \nabla_\beta(\delta\Gamma^{\alpha}{}_{\delta\beta}) - \nabla_\delta(\delta\Gamma^{\alpha}{}_{\beta\gamma})}$$

bona fide tensorial equation

(* if rather tedious ... ☺)

⊛ Now we can finally! write $\delta S_{grav}$ :-

(5.4) $$\delta S_{grav} = \frac{1}{16\pi G_N}\int d^Dx \left\{(\delta\sqrt{-g})\, g^{\alpha\beta}R_{\alpha\beta} + \sqrt{-g}\,(\delta g^{\alpha\beta})R_{\alpha\beta} + \sqrt{-g}\, g^{\alpha\beta}(\delta R_{\alpha\beta}) - 2\Lambda\,\delta(\sqrt{-g})\right\}$$

$\Rightarrow$

(5.5) $$\boxed{\begin{aligned}\delta S_{grav} = \frac{1}{16\pi G_N}\int d^Dx \Big\{ &\left(-\tfrac{1}{2}\sqrt{-g}\, g_{\mu\nu}R + \sqrt{-g}\, R_{\mu\nu}\right)\delta g^{\mu\nu} \\ &+ \Lambda\sqrt{-g}\, g_{\mu\nu}\,\delta g^{\mu\nu} \\ &+ \sqrt{-g}\, g^{\alpha\beta} g^{\gamma\delta}\left[\nabla_\gamma(\delta\Gamma^{\alpha}{}_{\delta\beta}) - \nabla_\delta(\delta\Gamma^{\alpha}{}_{\beta\gamma})\right]\Big\}\end{aligned}}$$

- Now, since

  (a) metric is compatible with connection

(6.1) $\Rightarrow \nabla_\alpha g_{\beta\gamma} = 0$

(6.2) ⊛ (b) for **any** vector $V^\beta$, $\boxed{\int d^Dx \sqrt{-g}\, \nabla_\mu V^\mu = \int d^Dx\, \partial_\mu V^\mu}$

(see Carroll and/or prove it for yourself)

i.e. get total derivative

= irrelevant on a topologically trivial spacetime (we will assume this here.)

(6.3) So
$$\boxed{\delta S_{grav} = \frac{1}{16\pi G_N} \int d^Dx \sqrt{-g} \left( R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right) \delta g^{\mu\nu}}$$

⊛ Now, typically $S_{matter}$ **does** involve $g^{\alpha\beta}$, even if only to contract up some vector or tensor indices to make a scalar! Otherwise, matter not coupled to gravity!

- So we derive that, with

(6.4) ⊛
$$\boxed{T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta (\mathcal{L}_{matter})}{\delta g^{\mu\nu}}}$$

(6.5)
$$\boxed{R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}}$$
⊛

VERY important equations; GR in a nutshell!