Action Principle for gravity

We can derive Einstein's equation $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R = 8\pi G_N T_{\mu\nu}$ from an action principle; this actually tells us how to define $T_{\mu\nu}$ in-principle.

What I'll do is propose an action for $g_{\mu\nu}$ that gives the Einstein equation, which is consistent with decent physical principles.

First, a reminder about Classical Field Theory

In classical mechanics, our variables were

$$q^a(t)$$

coordinates ↗ ↖ non-relativistic time

and the "velocities" were

$$\dot{q}^a(t) \quad , \qquad \dot{} \equiv d/dt .$$

We also defined canonical momenta

$$p_a \equiv \frac{\partial L}{\partial \dot{q}^a}$$

where the action

$$S = S[q^a(t)]$$
$$= \int dt \; L(q^a, \dot{q}^b)$$

and we can form the Hamiltonian

$$H = \sum_a p_a \dot{q}^a - L$$
$$= H(q^a, p_b) \qquad \text{(lives on } \underline{\text{phase}} \text{ } \underline{\text{space}}\text{)}$$

Hamilton's principle $\Leftrightarrow \delta S = 0$

Euler-Lagrange equations

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = 0$$

equivalent to Hamilton's equations

$$\frac{dp_a}{dt} = \{p_a, H\}$$
$$\frac{dq^a}{dt} = \{q^a, H\}$$

with $\{f, g\} = \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a} - \frac{\partial g}{\partial q^a}\frac{\partial f}{\partial p_a}$ (f, g live on phase space)

(In particular,
$$\{q^a, H\} = \frac{\partial H}{\partial p_a} \quad \text{and} \quad \{q^a, H\} = -\frac{\partial H}{\partial p_a}.$$

Field Theory: $\frac{d}{dt} \longrightarrow \partial_\mu$

and our "coordinates" are
$$q^a(t) \longrightarrow \phi^a(x^\mu)$$
Here, a is a collection of indices (could be a vector or tensor index, or an internal symmetry index (acted on by a gauge group, or something.).

The $\phi^a(x^\mu)$ are fields and in GR (i.e. physics!) they must be tensors.

Varying the action gives $\frac{\delta S[\phi^a(x^\mu)]}{\delta \phi^b} = 0$

gives, as you might expect,

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a}\right) = 0}$$

where

$$S[\phi^a(x^\mu)] = \int d^D x \, \mathcal{L}_{matter} = \int dt \, L$$

D (dim of spacetime); $\mathcal{L}_{matter}$ ← Lag. density; L ← Lagrangian

$$= \int (d^D x \sqrt{-g}) \left(\frac{1}{\sqrt{-g}} \mathcal{L}_{matter}\right)$$

[sometimes called $(e^{-1}\mathcal{L})$ by bein-savvy people :-]

Constructing action for $g_{\mu\nu}$

Note Title 13/10/2004

⊛ What set of principles will we use to construct our action for $g_{\mu\nu}$?

#1 : It should be invariant under coordinate transformations (perhaps up to a total derivative).

So if we write $S = \int d^D x \, \mathcal{L}$ we need to make sure (a) $\int d^D x \sqrt{-g}$ (etc.)

(b) (etc.) must be a scalar, not just a tensor.

$\Rightarrow \quad S = \int d^D x \sqrt{-g} \; (\text{scalar})$

⊛ What scalars can we build out of the metric tensor?

The easiest way I find to do this kind of thought process is by considering
- tensors first
- then contracting.

⊛ What do we have to work with? We've got

$$\nabla_\mu \,, \quad g_{\lambda\sigma} \,, \quad R_{\alpha\beta\gamma\delta} \,.$$

Can't contract ∇_μ with anything to make a scalar, because it has an odd # of indices.

One option is to consider

$$g^{\alpha\beta} g_{\alpha\beta} = g^{\alpha}{}_{\alpha} = D = \text{constant.}$$

Let's call $S_\Lambda = -2(\text{const.}) \int d^D x \sqrt{-g} \, \Lambda$

What about $\nabla^\mu \nabla^\nu g_{\mu\nu}$?

This is in-principle ok, but since we're using a metric-compatible connection so that $\nabla_\alpha g_{\mu\nu} = 0$ (which implies that $\nabla^\mu \nabla^\nu g_{\mu\nu} = 0$.)

So ... what else?

The next available thing is $R_{\alpha\beta\gamma\delta}\, g^{\alpha\gamma} g^{\beta\delta} = R$

This is the lowest-derivative-order invariant built out of ∇, g and R ... and then we would write

$$S_E = (\text{const.}) \int d^D x \sqrt{-g}\, R$$

Gives a well-defined initial-value problem because it involves only [up to] second-order derivatives of the fundamental dynamical variable $g_{\mu\nu}$.

So ... what is this (const.) ?
Being good physicists, we can work out what its dimensions must be, at least.

$g_{\mu\nu}$ itself has no dimensions because

$$\underbrace{ds^2}_{[\text{Length}]} = g_{\mu\nu} \underbrace{dx^\mu}_{[\text{Length}]} dx^\nu$$

So $\sqrt{-g}$ is dimensionless. Now, schematically,

$$R^{\cdot}{}_{\cdots} \sim \partial \Gamma^{\cdot}{}_{\cdot\cdot} + \Gamma^{\cdot}{}_{\cdot\cdot}\Gamma^{\cdot}{}_{\cdot\cdot} \quad (\text{etc.})$$

and

$$\Gamma^{\cdot}{}_{\cdot\cdot} \sim g^{\cdot\cdot}\, \partial_\cdot\, g_{\cdot\cdot} + (\text{etc.})$$

in other words, $R \sim 2$ derivatives on g i.e.

$$[R] = \frac{1}{\text{Length}^2} \equiv \frac{1}{L^2}$$

So, what of S_{grav}?

$$S_{grav.} = (\text{const.}) \int \underbrace{d^D x \sqrt{-g}\, R}$$

Units: $\hbar$ $\qquad L^{D-2}$

so $$[(\text{const.})] = L^{2-D}[\hbar]$$
$$= L^{2-D} \cdot M L T^{-1} L$$
$$= L^{4-D} M T^{-1}$$

Let $$(\text{const.}) = \frac{\hbar}{\ell_{Planck}^{D-2}} = \frac{\hbar}{\ell_p^{D-2}}$$

(*) Is this thingy related to G_N at all? To figure out if we're in the right ballpark, let's figure out the dimensions of G_N.

Potential for a point mass "$V = -\frac{GM}{r}$"

In D dimensions, Gauss's law ⇒ flux spreads out over $(D-2)$-sphere, rather than a 2-sphere, and the surface area for S^{D-2} goes like r^{D-2} ⇒

$$F \sim -\nabla V \sim -\frac{GMm}{r^{D-2}} \sim ma$$

i.e. $$[a] = \left[\frac{G_N M}{r^{D-2}}\right]$$

i.e. $$[G] = \left[\frac{r^{D-2} a}{M}\right]$$
$$= L^{D-2}(LT^{-2}) M^{-1}$$
$$= L^{D-1} T^{-2} M^{-1}$$

Let's compare this to
$$\left[\frac{\hbar}{\ell_p^{D-2}}\right] = L^{-D+2} M L T^{-1} L = L^{-D+4} M T^{-1}$$

So we're close; $[G^{-1}] = L^{1-D} M T^2$ ← we have right mass dimensions.

$\Rightarrow$ Try $[G^{-1} c^n] = L^{1-D} M T^2 (LT^{-1})^n$
$= L^{1+n-D} M T^{2-n}$

The dimensions match if $n=3$

so that $[G] = \left[\frac{\ell_p^{D-2} c^3}{\hbar}\right]$

(Note: particle physicist units : $\hbar = 1 = c$.)

It is conventional to set

$$\boxed{S_{grav} = \frac{1}{16\pi G_N} \int d^Dx \sqrt{-g}\,(R - 2\Lambda)}$$

and

$$S = S_{grav} + S_{matter}$$

⊕ The first step to finding the Einstein equation, which connects $\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}$ to $\underbrace{T_{\mu\nu}}$, is to compute $\frac{\delta S_{grav}}{\delta g^{\mu\nu}}$. We will later find that

$$\frac{\delta S_{grav}}{\delta g^{\mu\nu}} = -\frac{\delta S_{matter}}{\delta g^{\mu\nu}}$$

⊛ Let's vary S_{grav} w.r.t. $\underline{\underline{g^{\mu\nu}}}$ to find eqns of motion

✱ $\boxed{\text{"upstairs is easier"}}$

(c.f. upper-class British households with servants ... !)

So ... let's do it!

Want $\delta S = \delta \int d^Dx \sqrt{-g}\,(R - 2\Lambda)$ upon $\delta g^{\mu\nu}$.

How do we find $\delta\sqrt{-g}$?

- First of all : $g^\mu{}_\lambda = g^{\mu\alpha} g_{\alpha\lambda}$ and $\delta(g^\mu{}_\lambda) = 0$ so

δ^μ_λ