From Oct21 notes ... let's find Riemann sans 'bems ☺ ①

Note Title 26/10/2005

Prove that, in $d=2$, $G_{\mu\nu} \equiv 0$

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R}$$

Consider Euclidean coords

(1.1) $$\begin{cases} z = (x + i\tau)/\sqrt{2} \\ \bar{z} = (x - i\tau)/\sqrt{2} \end{cases}$$

(need $\tau \in \mathbb{R}$ for $\bar{z}$ to be z^*)

then flat spacetime is

(1.2) $$ds^2_{\text{Mink.}} = dz\, d\bar{z} + d\bar{z}\, dz \quad \text{i.e.} \quad \boxed{\eta_{z\bar{z}} = +1 = \eta_{\bar{z}z}}$$

In $d=2$, have $\frac{1}{2}(2)(3) = 3$ independent components of metric.
Also, have 2 coordinate transformation freedoms.
$\Rightarrow$ Any metric can $\therefore$ be put in the form (in $d=2$ only!)

$$ds^2_{(d=2)} = e^{2\phi(z,\bar{z})}(dz\, d\bar{z} + d\bar{z}\, dz)$$

(1.3) $$\Rightarrow \boxed{g_{z\bar{z}} = g_{\bar{z}z} = e^{2\phi}} \; ; \; g_{zz} = 0 \; ; \; g_{\bar{z}\bar{z}} = 0.$$

Christoffels

$$\Gamma^z{}_{zz} = \tfrac{1}{2} g^{z\bar{z}} (g_{z\bar{z},z} + g_{z\bar{z},z} - g_{zz,\bar{z}})$$
$$= g^{z\bar{z}} g_{z\bar{z},z} \qquad \because \text{only } g^{z\bar{z}} = e^{-2\phi} \text{ is nonzero}$$

Similarly,

(1.4) $$\Gamma^{\bar{z}}{}_{\bar{z}\bar{z}} = 2 g^{\bar{z}z} g_{\bar{z}z,\bar{z}}$$

$$\partial_z \equiv \partial \; ; \; \partial_{\bar{z}} \equiv \bar{\partial}$$

(1.5) i.e. $$\boxed{\Gamma^z{}_{zz} = 2\partial\phi} \; ; \; \boxed{\Gamma^{\bar{z}}{}_{\bar{z}\bar{z}} = 2\bar{\partial}\phi} \; ; \text{ others zero.}$$

Now recall our expression for Riemann (a real tensor) in terms of Christoffels and partial derivatives (not tensors):
Carroll (3.113)

(2.1) $$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}$$

(2.2) For our problem, we have $\{\partial_\alpha\} = \{\partial, \bar\partial\}$ and in "conformal gauge",

(2.3) where $ds^2 = (dz \otimes d\bar z + d\bar z \otimes dz)\cdot \exp(2\phi(z,\bar z))$,

we get

(2.4) $$\boxed{\Gamma^z{}_{zz} = 2\partial\phi(z,z)} \ \& \ \boxed{\Gamma^{\bar z}{}_{\bar z\bar z} = 2\bar\partial\phi(z,\bar z)}.$$

N.B. :

* our metric is off-diagonal (!) e.g. $g_{zz} = 0$

* Christoffels are purely diagonal e.g. $\Gamma^z{}_{z\bar z} = 0$

These two facts make computing Riemann much easier!

- In two dimensions, there is actually only one independent component of Riemann. Knowing symmetry properties of it, we can choose a representative to get rest from, like

(2.5) $$R_{z\bar z z\bar z} = g_{z\bar z} R^{\bar z}{}_{\bar z z\bar z} \quad \because \text{ off-diagonal metric}$$

$$= e^{2\phi}\left\{ \partial_z \Gamma^{\bar z}{}_{\bar z\bar z} - \partial_{\bar z} \underset{0}{\cancel{\Gamma^{\bar z}{}_{z\bar z}}} + \underset{0}{\cancel{\Gamma^{\bar z}{}_{z\lambda}}} \Gamma^\lambda{}_{\bar z\bar z} - \Gamma^{\bar z}{}_{\bar z\lambda} \underset{0}{\cancel{\Gamma^\lambda{}_{z\bar z}}} \right\}$$

(2.6) $$\therefore \boxed{R^{\bar z}{}_{\bar z z\bar z} = 2\partial\bar\partial\phi} \quad \Rightarrow \quad R^z{}_{z\bar z z} = 2\bar\partial\partial\phi = 2\partial\bar\partial\phi$$

Hence $R^{\bar z}{}_{\bar z \bar z z} = -2\partial\bar\partial\phi$ and $\Rightarrow R^{z}{}_{z z \bar z} = -2\partial\bar\partial\phi$

((34)-(43))

(2.7) but $R^{z}{}_{\bar z \alpha\beta} = 0 \because R_{(\mu\nu)\alpha\beta} = 0$ and g off-diagonal
and $R^{\bar z}{}_{z\alpha\beta} = 0$ too for mirror reasons.

$$\Rightarrow R_{zz} = R^{\lambda}{}_{z\lambda z} = R^{z}{}_{zzz} + R^{\bar z}{}_{z\bar z z} = 0 + 0 = 0 \qquad (2.8)$$

similarly, $R_{\bar z\bar z} = 0$. You might think $R_{\mu\nu} \equiv 0$, BUT !

Look at $R_{z\bar z} = R^{\lambda}{}_{z\lambda\bar z} = R^{z}{}_{zz\bar z} + R^{\bar z}{}_{z\bar z\bar z}$
$= -2\partial\bar\partial\phi + 0$
$= -2\partial\bar\partial\phi \neq 0$ (2.9)

similarly, $R_{\bar z z} = -2\partial\bar\partial\phi$

(2.10) $\Rightarrow R = g^{\mu\nu}R_{\mu\nu} = 2g^{z\bar z}R_{z\bar z} = -4e^{-2\phi}\partial\bar\partial\phi$

so $R_{z\bar z} - \frac{1}{2}g_{z\bar z}R = -2\partial\bar\partial\phi - \frac{1}{2}(e^{2\phi})(-4e^{-2\phi}\partial\bar\partial\phi)$

(2.11) $= 0$ (!) (& other components obviously zero too)

In other words, we've discovered that for ANY $\phi(z,\bar z)$
$R_{\mu\nu} \equiv \frac{1}{2}g_{\mu\nu}R$ in $D=2$ (ALWAYS)

(2.12) $\Rightarrow \boxed{G_{\mu\nu}(D=2) \equiv 0}$ QUOD ERAT DEMONSTRANDUM.

$$\boxed{\text{Riemann} \longrightarrow \text{Ricci scalar} + \text{Ricci tensor} + \text{Weyl tensor}}$$

Tracing Riemann?

- Contracting the first and third indices of (1,3) tensor Riemann gives Ricci tensor $R_{\mu\nu}$:

"contract 1 & 3 of (1,3) R" = mnemonic

(3.1) $$\boxed{R^{\lambda}{}_{\mu\lambda\nu} \equiv R_{\mu\nu}}$$

This Ricci tensor $R_{\mu\nu}$ is naturally
(3.2) symmetric as a (0, 2) tensor: $\boxed{R_{\nu\mu} = R_{\mu\nu}}$,
a property ensured by

(3.3) (a) $R_{\mu\nu\lambda\sigma} = R_{\lambda\sigma\mu\nu}$

(3.4) (b) $R_{\mu\nu\lambda\sigma} = -R_{\nu\mu\lambda\sigma}$

(3.5) (c) $R_{\mu\nu\lambda\sigma} = -R_{\mu\nu\sigma\lambda}$

(3.6) - Contracting again gives Ricci scalar R: $\boxed{R \equiv R^{\mu}{}_{\mu}}$

$(= g^{\mu\nu} R_{\mu\nu})$

⊛ What other data hide inside Riemann?

→ Life is more than just the traces you leave... ☺

How many independent components are there in $R^{\mu}{}_{\nu\lambda\sigma}$? Look for algebraic constraints.

→ use symmetry to determine # d.o.f. in Riemann

In principle, an unconstrained rank four tensor has

(4.1) $D \times D \times D \times D = D^4$ components.

e.g. $D=4$ (what we sense) $\Rightarrow 4^4 = 16 \times 16 = 256$!

Symmetries (a) – (c) on previous page mean that for d.o.f.-counting purposes we have a factorized structure:

(4.2) $$\frac{1}{2}\left\{\left[D(D-1)/2\right]\right\}\left\{\left[D(D-1)/2\right] + 1\right\}$$

sym $(12) \leftrightarrow (34)$

antisym $\begin{cases} (12) \leftrightarrow (21) \\ (34) \leftrightarrow (43) \end{cases}$

$$= \frac{1}{8}\left(D^4 - 2D^3 + 3D^2 - 2D\right)$$

• But there's more ! We also know that

(4.3) $$R_{[\alpha\beta\gamma\delta]} \equiv 0$$

This is

(4.4) $$\frac{1}{4!}(D)(D-1)(D-2)(D-3) \text{ equations in } D \text{ dimensions.}$$

(4.5) $\Rightarrow$ # cpts in Riemann $$n_{Riem} = \frac{1}{24}\left\{ 3\left[D(D-1)\right]\left[D^2 - D + 2\right] - D(D-1)\left[(D-2)(D-3)\right] \right\}$$

i.e.
$$n_{\text{Riem}} = \frac{1}{24} D(D-1)\left[3D^2 - 3D + 6 - D^2 + 5D - 6 \right]$$
$$= \frac{1}{12} D(D-1)\left[D(D+1) \right]$$

(5.1) $$\Rightarrow \quad \boxed{n_{\text{Riem}} = \frac{1}{12} D^2(D^2-1)}$$

- Ricci has obviously one degree of freedom (a real number at each point in spacetime).

- Ricci has $\frac{1}{2}D(D+1)$ components $\because$ symmetric rank $(0,2)$ so the remainder comes out to

(5.2) $$n_{\text{Weyl}} = \frac{1}{12} D(D+1)\, D(D-1) - \frac{1}{2} D(D+1)$$
$$= \frac{1}{12} D(D+1)\left[D(D-1) - 6 \right]$$

(5.3) $$\Rightarrow \quad \boxed{n_{\text{Weyl}} = \frac{1}{12} D(D+1)(D-3)(D+2)}$$

defined sensibly only when $\boxed{D \geq 4}$

Note: we don't do "$\frac{1}{4}$" for R, because R is already counted as part of $R_{\mu\nu}$.

(5.4) $$\boxed{C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{(D-2)}\left(g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho} \right) + \frac{2}{(D-2)(D-1)} g_{\rho[\mu} g_{\nu]\sigma} R}$$

Weyl Tensor Definition