GR — two dimensions in conformal gauge

Note Title 10/12/2004

Prove that, in $\boxed{d=2, \quad G_{\mu\nu}\equiv 0}$

⊛ $\boxed{\text{ZWEIBEIN METHOD}}$

Consider Euclidean coords

$$z = (x + i\tau)/\sqrt{2}$$
$$\bar{z} = (x - i\tau)/\sqrt{2}$$

(need $\tau \in \mathbb{R}$ for $\bar{z}$ to be z^*)

then flat spacetime is

$$ds^2_{\text{Mink.}} = dz\, d\bar{z} + d\bar{z}\, dz$$

In $d=2$, have $\frac{1}{2}(2)(3) = 3$ independent components of metric.
Also, have 2 coordinate transformation freedoms.
$\Rightarrow$ Any metric can $\therefore$ be put in the form (in $d=2$ only!)

$$ds^2_{(d=2)} = e^{2\phi(z,\bar{z})}(dz\, d\bar{z} + d\bar{z}\, dz)$$
$$\Rightarrow g_{z\bar{z}} = g_{\bar{z}z} = e^{2\phi} \;;\; g_{zz} = 0 \;;\; g_{\bar{z}\bar{z}} = 0.$$

(a) Obtain spin-connection from

$$g_{\mu\nu} = e^{\hat{a}}_{\mu} e^{\hat{b}}_{\nu} \eta_{\hat{a}\hat{b}}$$

and I take here $\begin{cases}\eta_{z\bar{z}} = +1 \\ \therefore \eta^{z\bar{z}} = +1\end{cases}$

- $\Rightarrow$ choose $e^{\hat{0}} = e^{\phi}\, dz \;;\quad e^{\hat{1}} = e^{\phi}\, d\bar{z}$

Then (e.g.) $g_{01} = e^{\hat{0}}_{0} e^{\hat{1}}_{1} \eta_{\hat{0}\hat{1}} + 0 = e^{\phi}.e^{\phi}.1 = e^{+2\phi}$

versus (e.g.) $g_{00} = e^{\hat{0}}_{0} e^{\hat{1}}_{0} \eta_{\hat{0}\hat{1}} + 0 = 0$

so

- $$de^{\hat{0}} = +\bar{\partial}\phi\, e^{\phi}\, d\bar{z}\wedge dz = -e^{-\phi}\bar{\partial}\phi\; e^{\hat{0}}\wedge e^{\hat{1}}$$
$$= -\omega^{\hat{0}}{}_{\hat{1}}\wedge e^{\hat{1}} - \omega^{\hat{0}}{}_{\hat{0}}\wedge e^{\hat{0}}$$

while

$$de^{\hat{1}} = \partial\phi\, e^{\phi}\, dz\wedge d\bar{z} = +e^{-\phi}\partial\phi\; e^{\hat{0}}\wedge e^{\hat{1}}$$
$$= -\omega^{\hat{1}}{}_{\hat{0}}\wedge e^{\hat{0}} - \omega^{\hat{1}}{}_{\hat{1}}\wedge e^{\hat{1}}$$

- How is $\omega^{A}{}_{B}$ related to $\omega^{B}{}_{A}$?

We have symmetry properties of two 'flat' tensors:

$$\omega_{\hat{A}\hat{B}} = -\omega_{\hat{B}\hat{A}} \quad \text{and} \quad \eta_{\hat{A}\hat{B}} = +\eta_{\hat{B}\hat{A}}$$

(This is true even though we have off-diagonal η.)

- We have $\omega_{AB} = \omega^{C}{}_{B}\,\eta_{CA}$

$$\Rightarrow \omega_{\hat{0}\hat{1}} = \omega^{\hat{0}}{}_{\hat{1}}\,\eta_{\hat{0}\hat{0}} + \omega^{\hat{1}}{}_{\hat{1}}\,\eta_{\hat{1}\hat{0}} = \omega^{\hat{1}}{}_{\hat{1}}$$
$$\text{and} \quad \omega_{\hat{1}\hat{0}} = \omega^{\hat{1}}{}_{\hat{0}}\,\eta_{\hat{1}\hat{1}} + \omega^{\hat{0}}{}_{\hat{0}}\,\eta_{\hat{1}\hat{0}} = \omega^{\hat{0}}{}_{\hat{0}}$$

So $\omega_{\hat{0}\hat{1}} = -\omega_{\hat{1}\hat{0}} \Rightarrow \omega^{\hat{1}}{}_{\hat{1}} = -\omega^{\hat{0}}{}_{\hat{0}}$ in our metric.

Also, $\omega^{\hat 1}{}_{\hat 0} = \omega^{\hat 1 \hat A}\eta_{\hat A \hat 0} = \omega^{\hat 1 \hat 1}\eta_{\hat 1 \hat 0} \equiv 0$ } in our
and $\omega^{\hat 0}{}_{\hat 1} = \omega^{\hat 0 \hat A}\eta_{\hat A \hat 1} = \omega^{\hat 0 \hat 0}\eta_{\hat 0 \hat 1} \equiv 0$ } metric

$\Rightarrow$ $de^{\hat 0} = -\omega^{\hat 0}{}_{\hat 0}\wedge e^{\hat 0}$ and $de^{\hat 1} = -\omega^{\hat 1}{}_{\hat 1}\wedge e^{\hat 1}$
while $de^{\hat 0} = -e^{-\phi}\bar\partial\phi\, e^{\hat 0}\wedge e^{\hat 1}$ & $de^{\hat 1} = +e^{-\phi}\partial\phi\, e^{\hat 0}\wedge e^{\hat 1}$

$\Rightarrow$ $+\omega^{\hat 0}{}_{\hat 0}\wedge e^{\hat 0} = +e^{-\phi}\bar\partial\phi\, e^{\hat 0}\wedge e^{\hat 1}$
Let $\omega^{\hat 0}{}_{\hat 0} := f_0 e^{\hat 0} + f_1 e^{\hat 1} \Rightarrow \omega^{\hat 0}{}_{\hat 0}\wedge e^{\hat 0} = f_1 e^{\hat 1}\wedge e^{\hat 0} = -f_1 e^{\hat 0}\wedge e^{\hat 1}$
$\Rightarrow f_1 = +e^{-\phi}\bar\partial\phi$
Similarly, let $\omega^{\hat 1}{}_{\hat 1} = g_0 e^{\hat 0} + g_1 e^{\hat 1} \Rightarrow \omega^{\hat 1}{}_{\hat 1}\wedge e^{\hat 1} = g_0\, e^{\hat 0}\wedge e^{\hat 1}$
$\Rightarrow g_0 = -e^{-\phi}\partial\phi$

- Now we use antisymmetry of ω_{AB} ($\omega^{\hat 0}{}_{\hat 0} = -\omega^{\hat 1}{}_{\hat 1}$)

$-\omega^{\hat 0}{}_{\hat 0} = f_0 e^{\hat 0} + e^{-\phi}\bar\partial\phi\, e^{\hat 1} = \omega^{\hat 1}{}_{\hat 1} = -e^{-\phi}\partial\phi\, e^{\hat 0} + g_1 e^{\hat 1}$
$\Rightarrow f_0 = -e^{-\phi}\partial\phi$ and $g_1 = e^{-\phi}\bar\partial\phi$
Thus, $-\omega^{\hat 0}{}_{\hat 0} = +\omega^{\hat 1}{}_{\hat 1} = e^{-\phi}[-\partial\phi\, e^{\hat 0} + \bar\partial\phi\, e^{\hat 1}]$

i.e. $\boxed{-\omega^{\hat 0}{}_{\hat 0} = e^{-\phi}\left[\bar\partial\phi\, e^{\hat 1} - \partial\phi\, e^{\hat 0}\right] = \omega^{\hat 1}{}_{\hat 1}}$ & $\omega^{\hat 0}{}_{\hat 1} = 0$

Alternatively, $\boxed{-\omega^{\hat 0}{}_{\hat 0} = \bar\partial\phi\, d\bar z - \partial\phi\, dz = \omega^{\hat 1}{}_{\hat 1}}$ & $\omega^{\hat 1}{}_{\hat 0} \equiv 0$

So $\omega^{\hat 0}{}_{\hat 0} = \partial\phi\, dz - \bar\partial\phi\, d\bar z$ and $\omega^{\hat 1}{}_{\hat 1} = \bar\partial\phi\, d\bar z - \partial\phi\, dz$.

Therefore,

- $R^{\hat 0}{}_{\hat 0} = -\partial\bar\partial\phi\, dz\wedge d\bar z - \bar\partial\partial\phi\, dz\wedge d\bar z + (\bar\partial\phi\, d\bar z - \partial\phi\, dz)\wedge(\bar\partial\phi\, d\bar z - \partial\phi\, dz)$

$= -2\partial\bar\partial\phi\, dz\wedge d\bar z$

$\Rightarrow$ $\boxed{R^{\hat 0}{}_{\hat 0 \hat 0 \hat 1} = -4\partial\bar\partial\phi}$ ☺ agrees with grtensor.

By symmetry, $\boxed{R^{\hat 1}{}_{\hat 1 \hat 1 \hat 0} = -4\bar\partial\partial\phi}$

- Contracting to find Ricci, get $\boxed{R_{\hat 0 \hat 1} = -8\partial\bar\partial\phi}$

so Ricci scalar $\boxed{R = -16\partial\bar\partial\phi}$
$\Rightarrow$ $G_{\hat 0 \hat 1} \equiv 0$. (All other components zero.)

$$* \quad \tilde{R}_{\sigma\nu} = R_{\sigma\nu} - \omega^{-1}\nabla_\alpha\nabla_\beta\omega\left[(n-2)g^\alpha_\nu g^\beta_\sigma + g_{\sigma\nu}g^{\alpha\beta}\right]$$
$$+ \omega^{-2}\nabla_\alpha\omega\nabla_\beta\omega\left[2(n-2)g^\alpha_\nu g^\beta_\sigma - (n-3)g_{\sigma\nu}g^{\alpha\beta}\right]$$

ex HW#2

$$\bullet \quad \tilde{R} = \omega^{-2}R - 2(n-1)\omega^{-3}\nabla^2\omega - (n-1)(n-4)\omega^{-4}(\nabla\omega)^2$$

$$\therefore \quad \tilde{R}_{\sigma\nu} - \tfrac{1}{2}\tilde{g}_{\sigma\nu}\tilde{R} = \tilde{G}_{\mu\nu}$$
$$= G_{\sigma\nu} + \left\{-\omega^{-1}\nabla_\alpha\nabla_\beta\omega\left[g_{\sigma\nu}g^{\alpha\beta}\right] - \omega^{-2}\nabla_\alpha\omega\nabla_\beta\omega\left[-g_{\sigma\nu}g^{\alpha\beta}\right]\right\}$$
$$- \frac{1}{2}\left[-2\omega^{-1}\nabla^2\omega + 2\omega^{-2}(\nabla\omega)^2\right]g_{\sigma\nu}$$

so that $\tilde{G}_{\sigma\nu} = G_{\sigma\nu} + g_{\sigma\nu}\left[\begin{array}{l}-\omega^{-1}\nabla^2\omega + \omega^{-2}(\nabla\omega)^2 \\ +\omega^{-1}\nabla^2\omega - \omega^{-2}(\nabla\omega)^2\end{array}\right]$

$\tilde{G}_{\sigma\nu} = G_{\sigma\nu}$ ☺

(c) Killing vectors solve

$$\nabla_{(\mu}K_{\nu)} = 0 = \partial_{(\mu}K_{\nu)} - \Gamma^\sigma_{\mu\nu}K_\sigma$$

$\Rightarrow$ need Christoffel symbols

Christoffels?

$$\Gamma^z{}_{zz} = \tfrac{1}{2}g^{z\bar{z}}\left(g_{z\bar{z},z} + g_{z\bar{z},z} - g_{zz,\bar{z}}\right)$$
$$= g^{z\bar{z}}\,g_{z\bar{z},z} \qquad \because \text{ only } g^{z\bar{z}} = e^{-2\phi} \text{ is nonzero}$$

Similarly,

$$\Gamma^{\bar{z}}{}_{\bar{z}\bar{z}} = 2g^{\bar{z}z}\,g_{\bar{z}z,\bar{z}} \qquad \boxed{\partial_z \equiv \partial \,;\, \partial_{\bar{z}} \equiv \bar{\partial}}$$

i.e. $\Gamma^z{}_{zz} = 2\partial\phi$; $\Gamma^{\bar{z}}{}_{\bar{z}\bar{z}} = 2\bar{\partial}\phi$; others zero.]

Only nonzero Γ's are $\Gamma^1{}_{11}$ and $\Gamma^0{}_{00}$ ...

$$\Rightarrow \quad \partial_z K_z - 2\partial_z\phi\, K_z = 0 \qquad (zz \text{ eqn})$$

$$\Rightarrow \quad K_z = c_1 e^{2\phi} \quad \text{i.e.} \quad K^{\bar{z}} = c_1 \qquad c_1 = 1 \Rightarrow \text{normalized to unity.}$$

$\bar{z}z$ eqn is

$$\nabla_{\bar{z}}K_z = \partial_{\bar{z}}K_z = \bar{\partial}K_z = 0 \text{ iff } \boxed{\phi = \phi(z \text{ only}) \leftrightarrow K^{\bar{z}} = 1}$$

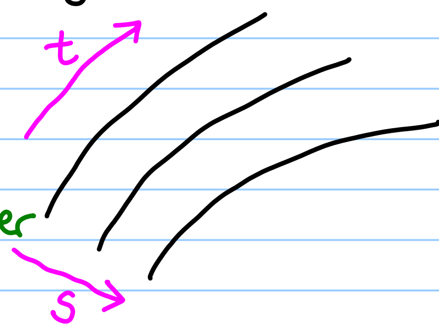
$z\bar{z}$ eqn is trivial; $K_{\bar{z}} = 0$.

Similarly, then $K_{\bar{z}} = c_2 e^{2\phi}$ i.e. $K^z = c_2$ $\quad c_2 = 1 \Rightarrow$ nomalized

$$\partial K_{\bar{z}} = 0 \text{ iff } \boxed{\phi = \phi(\bar{z} \text{ only}) \leftrightarrow K^z = 1} \text{ etc.}$$

Geodesic Deviation

Suppose we have a family of geodesics that don't cross.



Tangent vector to geodesics

$$T^\mu = \frac{\partial x^\mu}{\partial t}$$ ← affine parameter

Deviation vector

$$S^\mu = \frac{\partial x^\mu}{\partial s}$$

- "Relative velocity" of geodesics (definition)

$$V^\mu = (\nabla_T S)^\mu$$

(1.1) $$= T^\rho \nabla_\rho S^\mu$$ (a tensor ✓)

- "Relative acceleration" of geodesics (definition)

$$A^\mu = (\nabla_T V)^\mu$$

(1.2) $$= T^\rho \nabla_\rho V^\mu = T^\rho \nabla_\rho (T^\sigma \nabla_\sigma) S^\mu$$

- S & T were carefully picked to be adapted to the above coord system:

(1.3) $$[S, T] = 0.$$

This implies

(1.4) $$S^\rho \nabla_\rho T^\mu = T^\rho \nabla_\rho S^\mu.$$

Then (algebra on Carroll p146)

(1.5) $$\boxed{A^\mu = \frac{D^2}{dt^2} S^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma}$$

This is called the <u>geodesic deviation equation</u>.

⇒ <u>Riemann is responsible for tidal forces!</u>
(e.g.: this makes astronaut <u>very</u> uncomfy crossing $M_\odot$ BH!)