2005-10-19 : 'Beins continued ; and Riemann tensor (1)

(1.1) Basic eqn : $g_{\mu\nu} = e^A{}_\mu e^B{}_\nu \eta_{AB}$.

Of course, the choice of 'beins is not unique... So let's pick really simply.

(1.2a) Define $\boxed{e^{\hat{0}}{}_0 = c}$

and

(1.2b) $\boxed{e^{\hat{i}}{}_j = a(t)\, \delta^{\hat{i}}{}_j}$

Then we easily reproduce (5.1) ☺

(1.3) ⊛ Now look at $de^A + \omega^A{}_B \wedge e^B = 0$.

First step : form $e^{\hat{0}} = e^{\hat{0}}{}_\alpha dx^\alpha = c\,dt$
and $e^{\hat{i}} = e^{\hat{i}}{}_\alpha dx^\alpha = a(t)\, dx^i$

(1.4) Then $de^{\hat{0}} = d(c\,dt) = 0$
$= -\omega^{\hat{0}}{}_{\hat{0}} \wedge e^{\hat{0}} - \omega^{\hat{0}}{}_{\hat{i}} \wedge e^{\hat{i}}$

Now we need to know a useful property of $\omega^A{}_B$:

(1.5) $$\boxed{\omega_{AB} = \omega_{[AB]}}$$

Then, using $\eta^{\hat{A}\hat{B}}$ to raise, etc., we have

$$\omega^{\hat{0}}{}_{\hat{0}} = \omega^{\hat{0}\hat{\delta}}\eta_{\hat{\delta}\hat{0}} + 0$$

(2.1) $$= -\omega^{\hat{0}\hat{0}} = 0 \quad \text{by antisymmetry.}$$

(2.2) Also, $\omega^{\hat{0}}{}_{\hat{\imath}} = \eta^{\hat{0}\hat{\delta}}\omega_{\hat{\delta}\hat{\imath}} = -\eta^{\hat{0}\hat{0}}\omega_{\hat{\imath}\hat{0}} = -\omega_{\hat{\imath}}{}^{\hat{0}}$ (!)

So $de^{\hat{0}} = 0 \Rightarrow \omega^{\hat{0}}{}_{\hat{\imath}} \wedge e^{\hat{\imath}} = 0$

(2.3) but $e^{\hat{\imath}} = a(t)\, dx^i \Rightarrow \boxed{\omega^{\hat{0}}{}_{\hat{\imath}} = \alpha(t)\, e^{\hat{\imath}}}$ $\exists\, \alpha(t)$

(we'll find this shortly)

• Spatial eqns? Have $de^{\hat{\imath}} = d(a(t)\, dx^i)$

$$= \dot{a}\, dt \wedge dx^i = \left(\frac{1}{c}\frac{\dot{a}}{a}\right) e^{\hat{0}} \wedge e^{\hat{\imath}}$$

(2.4) $$= -\omega^{\hat{\imath}}{}_{\hat{0}} \wedge e^{\hat{0}} - \omega^{\hat{\imath}}{}_{\hat{k}} \wedge e^{\hat{k}}$$

$$\Rightarrow +\omega^{\hat{0}}{}_{\hat{\imath}} \wedge e^{\hat{0}} = \left(\frac{1}{c}\frac{\dot{a}}{a}\right) e^{\hat{\imath}} \wedge e^{\hat{0}}$$

(2.5) i.e. $$\boxed{\omega^{\hat{0}}{}_{\hat{\imath}} = \frac{1}{c}\frac{\dot{a}}{a}\, e^{\hat{\imath}}}$$ ☺ Q.E.D.

$$= \tfrac{1}{c}\, \dot{a}\, dx^i$$

For our last trick, let's find Riemann.

Starting from our choice $\begin{cases} e^{\hat{0}} = c\,dt \\ e^{\hat{i}} = a(t)\,dx^i \end{cases}$

we got
$$\begin{cases} \omega_{\hat{0}\hat{0}} \equiv 0 & (3.1a) \\ \omega_{\hat{0}\hat{i}} = -\omega_{\hat{i}\hat{0}} = -\frac{1}{c}\frac{\dot{a}}{a} e^{\hat{i}} & (3.1b) \\ \omega_{\hat{i}\hat{j}} = 0 & (3.1c) \end{cases}$$

$\Rightarrow R^{\hat{0}}{}_{\hat{0}} = \omega^{\hat{0}}{}_{\hat{i}} \wedge \omega^{\hat{i}}{}_{\hat{0}}$ by antisymmetry of ω_{AB}

(3.2) $= 0$ by property of ω & of $\wedge$

Also, $R^{\hat{0}}{}_{\hat{i}} = d\omega^{\hat{0}}{}_{\hat{i}} + \omega^{\hat{0}}{}_{\hat{j}} \wedge \omega^{\hat{j}}{}_{\hat{i}}$

(3.3) $= d\left(\frac{1}{c}\dot{a}\,dx^i\right) = \frac{1}{c}\ddot{a}\,dt \wedge dx^i = \frac{1}{c^2}\frac{\ddot{a}}{a} e^{\hat{0}} \wedge e^{\hat{i}}$

(3.4) Then $R^{\hat{i}}{}_{\hat{0}} = \frac{1}{c^2}\frac{\ddot{a}}{a} e^{\hat{0}} \wedge e^{\hat{i}}$

(3.5) Lastly, $R^{\hat{i}}{}_{\hat{j}} = \omega^{\hat{i}}{}_{\hat{0}} \wedge \omega^{\hat{0}}{}_{\hat{j}} = -\frac{1}{c^2}\left(\frac{\dot{a}}{a}\right)^2 e^{\hat{i}} \wedge e^{\hat{j}}$

(3.6a) $\Rightarrow$ $\boxed{R^{\hat{0}}{}_{\hat{i}\hat{0}\hat{j}} = \frac{\ddot{a}}{a}\delta_{\hat{i}\hat{j}}}$ $\boxed{R^{\hat{i}}{}_{\hat{0}\hat{0}\hat{j}} = \frac{\ddot{a}}{a}\delta^{\hat{i}}{}_{\hat{j}}}$

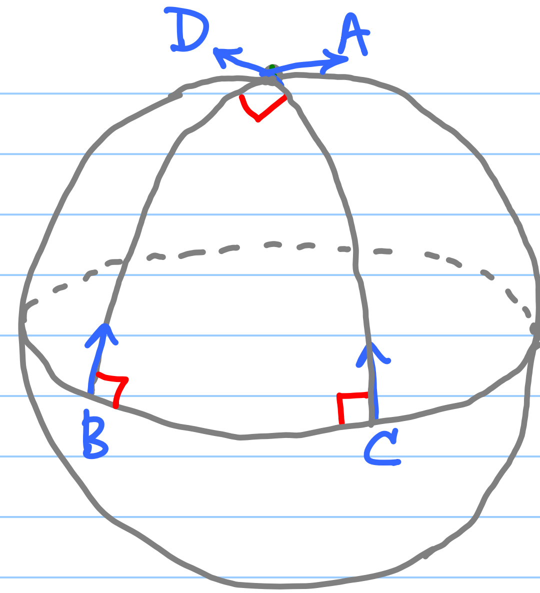
(3.6b)

(3.6c) $\boxed{R^{\hat{i}}{}_{\hat{j}\hat{k}\hat{l}} = \left(\frac{\dot{a}}{a}\right)^2 \left(\delta^{\hat{i}}{}_{\hat{k}}\delta_{\hat{j}\hat{l}} - \delta^{\hat{i}}{}_{\hat{l}}\delta_{\hat{j}\hat{k}}\right)}$

- Full contraction: $R^{\hat{A}}{}_{\hat{B}\hat{A}}{}^{\hat{B}} = 2(d-1)\frac{\ddot{a}}{a} + (d-1)(d-2)\left(\frac{\dot{a}}{a}\right)^2.$ (3.7)

- Singular where $a(t) \to 0$ (or $\ddot{a}$ or $\dot{a} \to \infty$) "Big Bang/Crunch".

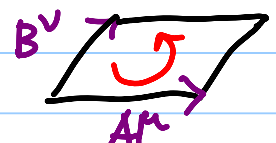
Curvature



- For our S^2 example last time, we discovered that parallel transporting a vector around a closed loop does <u>not</u> necessarily return it to its original state.

(*) The extent to which this fails is called <u>curvature</u>.

Equations? Consider an infinitesimal loop



non-zero area if A not // to B.

Under parallel-transport around the loop we get a change in V, δV, which is also a vector; in order to express δV in terms of V, A, B we need a (1,3) tensor:

$$\delta V^\mu = R^\mu{}_{\nu\alpha\beta} V^\nu A^\alpha B^\beta \tag{4.1}$$

- Another way to define curvature is via the commutator of covariant derivatives:

(5.1)
$$\boxed{[\nabla_\mu, \nabla_\nu] V^\alpha = R^\alpha{}_{\beta\mu\nu} V^\beta}$$ (for a torsion-free connection)

It can be written in terms of "$\partial\Gamma$" and "$\Gamma\times\Gamma$":

(5.2)
$$\boxed{R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha{}_{\nu\beta} - \partial_\nu \Gamma^\alpha{}_{\mu\beta} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\gamma{}_{\nu\beta} - \Gamma^\alpha{}_{\nu\gamma}\Gamma^\gamma{}_{\mu\beta}}$$

(5.3) Notice that $\boxed{R^\alpha{}_{\beta\mu\nu} = -R^\alpha{}_{\beta\nu\mu}}$

which encodes the fact that traversing the loop in the above figure in the opposite direction gives the 'opposite' answer i.e. with a minus sign (area element is oriented oppositely.)

⊛ On a general tensor (for torsion-free connection)

(5.4)
$$\boxed{\begin{aligned}[\nabla_\alpha, \nabla_\beta] X^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} = {} & R^{\mu_1}{}_{\gamma\alpha\beta} X^{\gamma\mu_2\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} \\ & + R^{\mu_2}{}_{\gamma\alpha\beta} X^{\mu_1\gamma\mu_3\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} + \cdots \\ & - R^{\gamma}{}_{\nu_1\alpha\beta} X^{\mu_1\cdots\mu_k}{}_{\gamma\nu_2\cdots\nu_\ell} \\ & - R^{\gamma}{}_{\nu_2\alpha\beta} X^{\mu_1\cdots\mu_k}{}_{\nu_1\gamma\nu_3\cdots\nu_\ell} - \cdots\end{aligned}}$$

⊛ Symmetry properties of Riemann:-

(6.1) $$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$$

(6.2) $$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$$

(6.3) $$R_{\rho[\sigma\mu\nu]} = 0$$

(6.4) $$R_{[\rho\sigma\mu\nu]} = 0$$

(6.5) $$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$$

Ricci tensor, Ricci scalar, Einstein tensor

(6.6) $$R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu}$$

(6.7) $$R \equiv R^{\mu}{}_{\mu}$$

More on this next time! ☺

(6.8) $$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Then

(6.9) $$\nabla^{\mu} G_{\mu\nu} = 0$$

← this is the germ of the gravity part of Einstein's famous equation of General Relativity.

→ more later ☺