"Beins" ☺

- It is really important (mathematically and physically) to keep in mind that the spacetime manifold — while typically curved and lumpy — is locally flat spacetime.

(1.1a) Let flat-type indices be $\boxed{\{A\}}$, $A = 0, 1, 2$ or 3

(1.1b) & curved-type indices be $\boxed{\{\alpha\}}$, $\alpha = 0, 1, 2$ or 3

⊛ At each point, since spacetime is locally flat, we can write for (say) a (1,0) tensor V

$$\boxed{V^A = e^A{}_\mu V^\mu}$$

"vierbein" in 4-d

invertible, 4^2 components

Similarly:-
(0,1) tensors have flat components

$$\boxed{\omega_A = e_A{}^\mu \omega_\mu}$$

where

$$\boxed{e_A{}^\mu e^B{}_\mu = \delta_A{}^B} \quad \& \quad \boxed{e_A{}^\mu e^A{}_\nu = \delta^\mu{}_\nu}$$

$= \eta_A{}^B$ $\qquad$ $= g^\mu{}_\nu$

⊛ Flat indices are raised/lowered via η^{AB} & η_{CD};

⊛ curved indices are raised/lowered via $g^{\mu\nu}$ & $g_{\alpha\beta}$.

- Since g is a rank $(0,2)$ tensor, we naturally write

$$\boxed{g_{\mu\nu} = e^A{}_\mu \, e^B{}_\nu \, \eta_{AB}}$$

Spin connection 1-form $\omega^A{}_B$

- The Christoffel is not the only available connection allowing us to parallel-transport stuff around...! In particular, $\Gamma^\alpha{}_{\beta\gamma}$ does not correctly provide for parallel-transport of SPINORS
  ↳ objects with spin $\frac{1}{2}(2n+1)\hbar$, $n \in \mathbb{Z}^+$.
  (e.g. electron!)

⊛ Spin connection 1-form

$$\boxed{\omega^A{}_B = \omega^A{}_{B\mu} \, dx^\mu}$$

(components in coördinate basis)

- [We set torsion to zero, so] the equation satisfied by $\omega^A{}_B$ is

$$\boxed{de^A + \omega^A{}_B \wedge e^B = 0}$$

This is a 2-form equation because (e.g.)

$e^B = e^B{}_\mu dx^\mu$ is a 1-form

and $\omega^A{}_B = \omega^A{}_{B\mu} dx^\mu$ is a 1-form too.

⊛ The equation above is usually the quickest way to find $\omega^A{}_B$! ☺

- Spinors carry special representations of $SO(1,3)$, the Lorentz group.

For a spinor (with spinor components ψ^α), ω defines

$$\hat{\nabla}_\mu \psi = \left(\partial_\mu + \frac{1}{4}\omega_{AB\mu}\Gamma^{[AB]}\right)\psi$$

This is useful technology for those interested in particle theory and string theory !!

Riemann two-form $R^A{}_B$

- The spin connection is also useful because it leads directly to the Riemann two-form

(4.1) $$\boxed{R^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B}$$

We will have a <u>lot</u> more to say about Riemann in the coming weeks. For now, just treat this as a mathematical definition of an object whose physical significance we have yet to elucidate and understand... ☺

- Let's illustrate the power of the language of differential forms by doing an example of how to compute vierbeins using our equation

(4.2) $$de^A + \omega^A{}_B \wedge e^B = 0$$

which has best friend (4.3), i.e.

$$R^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B$$

} "Cartan structure equations"

(4.3) Let's do the example $ds^2 = -c^2dt^2 + a^2(t)\,|d\vec{x}|^2$

- In (4.3), all spatial distances scaled by $a(t)$ → "SCALE FACTOR"

(5.1) Basic eqn : $$g_{\mu\nu} = e^A{}_\mu e^B{}_\nu \eta_{AB} .$$

Of course, the choice of 'beins is not unique ...
So let's pick really simply.

(5.2a) Define $\boxed{e^{\hat{0}}{}_0 = c}$
and
(5.2b) $\boxed{e^{\hat{i}}{}_j = a(t)\, \delta^{\hat{i}}{}_j}$

Then we easily reproduce (5.1) ☺

⊛ Now look at $de^A + \omega^A{}_B \wedge e^B = 0$.

First step : form $e^{\hat{0}} = e^{\hat{0}}{}_\alpha dx^\alpha = c\,dt$
and $e^{\hat{i}} = e^{\hat{i}}{}_\alpha dx^\alpha = a(t)\, dx^i$

Then $$de^{\hat{0}} = d(c\,dt) = 0$$
$$= -\omega^{\hat{0}}{}_{\hat{0}} \wedge e^{\hat{0}} - \omega^{\hat{0}}{}_{\hat{i}} \wedge e^{\hat{i}}$$

Now we need to know a useful property of $\omega^A{}_B$:

$$\boxed{\omega_{AB} = \omega_{[AB]}}$$

Then, using $\eta^{\hat{A}\hat{B}}$ to raise, etc., we have

$$\omega^{\hat{0}}{}_{\hat{0}} = \omega^{\hat{0}\hat{0}}\eta_{\hat{0}\hat{0}} + 0$$

$$= -\omega^{\hat{0}\hat{0}} = 0 \quad \text{by antisymmetry.}$$

Also, $\omega^{\hat{0}}{}_{\hat{\imath}} = \eta^{\hat{0}\hat{0}}\omega_{\hat{0}\hat{\imath}} = -\eta^{\hat{0}\hat{0}}\omega_{\hat{\imath}\hat{0}} = -\omega_{\hat{\imath}}{}^{\hat{0}}$ (!)

So $de^{\hat{0}} = 0 \Rightarrow \omega^{\hat{0}}{}_{\hat{\imath}} \wedge e^{\hat{\imath}} = 0$

but $e^{\hat{\imath}} = a(t)\,dx^i \Rightarrow \omega^{\hat{0}}{}_{\hat{\imath}} = \alpha(t)\,e^{\hat{\imath}}$ $\exists\,\alpha(t)$

(we'll find this shortly)

- Spatial eqns? Have

$$de^{\hat{\imath}} = d\left(a(t)\,dx^i\right)$$

$$= \dot{a}\,dt \wedge dx^i = \left(\frac{1}{c}\frac{\dot{a}}{a}\right) e^{\hat{0}} \wedge e^{\hat{\imath}}$$

$$= -\omega^{\hat{\imath}}{}_{\hat{0}} \wedge e^{\hat{0}} - \omega^{\hat{\imath}}{}_{\hat{k}} \wedge e^{\hat{k}}$$

$$\Rightarrow +\omega^{\hat{0}}{}_{\hat{\imath}} \wedge e^{\hat{0}} = \left(\frac{1}{c}\frac{\dot{a}}{a}\right) e^{\hat{\imath}} \wedge e^{\hat{0}}$$

i.e. $\omega^{\hat{0}}{}_{\hat{\imath}} = \dfrac{1}{c}\dfrac{\dot{a}}{a}\,e^{\hat{\imath}}$

$$= 1/c\,\dot{a}\,dx^i$$

Q.E.D.

For our last trick, let's find Riemann.

Starting from our choice $\begin{cases} e^{\hat{0}} = c\,dt \\ e^{\hat{i}} = a(t)\,dx^i \end{cases}$

we got $\begin{cases} \omega_{\hat{0}\hat{0}} \equiv 0 \\ \omega_{\hat{0}\hat{i}} = -\omega_{\hat{i}\hat{0}} = -\frac{1}{c}\frac{\dot{a}}{a} e^{\hat{i}} \\ \omega_{\hat{i}\hat{j}} = 0 \end{cases}$

$$\Rightarrow R^{\hat{0}}{}_{\hat{0}} = \omega^{\hat{0}}{}_{\hat{i}} \wedge \omega^{\hat{i}}{}_{\hat{0}} \quad \text{by antisymmetry of } \omega_{AB}$$
$$= 0 \quad \text{by property of } \omega \text{ \& of } \wedge$$

Also, $R^{\hat{0}}{}_{\hat{i}} = d\omega^{\hat{0}}{}_{\hat{i}} + \omega^{\hat{0}}{}_{\hat{j}} \wedge \omega^{\hat{j}}{}_{\hat{i}}$
$$= d\left(\frac{1}{c}\dot{a}\,dx^i\right) = \frac{1}{c}\ddot{a}\,dt \wedge dx^i = \frac{1}{c^2}\frac{\ddot{a}}{a} e^{\hat{0}} \wedge e^{\hat{i}}$$

Then $R^{\hat{i}}{}_{\hat{0}} = \frac{1}{c^2}\frac{\ddot{a}}{a} e^{\hat{0}} \wedge e^{\hat{i}}$

Lastly, $R^{\hat{i}}{}_{\hat{j}} = \omega^{\hat{i}}{}_{\hat{0}} \wedge \omega^{\hat{0}}{}_{\hat{j}} = -\frac{1}{c^2}\left(\frac{\dot{a}}{a}\right)^2 e^{\hat{i}} \wedge e^{\hat{j}}$

$$\Rightarrow R^{\hat{0}}{}_{\hat{i}\hat{0}\hat{j}} = \frac{\ddot{a}}{a}\delta_{\hat{i}\hat{j}} \qquad R^{\hat{i}}{}_{\hat{0}\hat{0}\hat{j}} = \frac{\ddot{a}}{a}\delta^{\hat{i}}{}_{\hat{j}}$$
$$R^{\hat{i}}{}_{\hat{j}\hat{k}\hat{l}} = \left(\frac{\dot{a}}{a}\right)^2 \left(\delta^{\hat{i}}{}_{\hat{k}}\delta_{\hat{j}\hat{l}} - \delta^{\hat{i}}{}_{\hat{l}}\delta_{\hat{j}\hat{k}}\right)$$

- Full contraction: $R^{\hat{A}}{}_{\hat{B}\hat{A}}{}^{\hat{B}} = 2(d-1)\frac{\ddot{a}}{a} + (d-1)(d-2)\left(\frac{\dot{a}}{a}\right)^2$.

- Singular where $a(t) \to 0$ (or $\ddot{a}$ or $\dot{a} \to \infty$) "Big Bang/Crunch".