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Note Title 11/10/2005

Today:

(a) Commutator $[X,Y]$ of 2 vector fields (1/6)

(b) Lie Derivative $\mathcal{L}_V$ (1/2)

(c) Killing tensors & symmetries (1/3)

& a couple of simple examples

On Friday:

- vierbeins e^A and spin connection ω^{AB}; "Cartan structure equations"

$$de^A + \omega^A{}_B \wedge e^B = 0$$

$$d\omega^{AB} + \omega^A{}_C \wedge \omega^{CB} = R^{AB}$$

curvature of spacetime

Professor Erich Poppitz will sub. for me

(holder of 1st PGSA teaching award! ☺)

Commutator for 2 vector fields

- Define <u>commutator</u> of 2 vector fields X and Y by

(1.1) $$[X,Y](f) \equiv X(Y(f)) - Y(X(f)) \qquad \exists f(x^\lambda)$$

- The neat thing about this guy $[X,Y]$ is that it is a bona fide vector field: it

(a) is linear

(1.2) $$[X,Y](af+bg) = a[X,Y]f + b[X,Y]g$$

and

(b) obeys the Leibniz rule

(1.3) $$[X,Y](fg) = f[X,Y]g + g[X,Y]f$$

- In components (in the coordinate basis), the new vector field $[X,Y]$ has components

(1.4) $$\boxed{[X,Y]^\mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu}$$

⊛ This <u>is</u> a well-defined tensor: the non-tensorial pieces from the partial derivatives CANCEL by antisymmetry of $[X,Y]$ under $X \leftrightarrow Y$ ☺ !

Lie Derivative

(Pronunciation note: "Lie" rhymes with "see")

- A more general construction is called the Lie derivative. This is arguably a more primitive concept than our covariant derivative ∇... !

- We start by defining the integral curves of a vector field $V(x)$ to be those curves $x^\mu(t)$ satisfying

(2.1) $$\boxed{\frac{dx^\mu}{dt} = V^\mu}$$

logic: { given V, find its integral curves.

▸ A familiar example: magnetic flux lines are the integral curves of $\vec{B}$.

- It's interesting to ask
  [How fast does a tensor T change along integral curves of V?

  ⊛ This is the Lie derivative of T along V, $\mathcal{L}_V(T)$.

To illustrate, let's pick special coordinates

(3.1a) Picking $\boxed{\{x^\mu\} = \{x^0, x^1, x^2, x^3\}}$

the parameter along integral curves of V

(3.1b) $\Rightarrow V = \frac{\partial}{\partial x^1}$ in this special coördinate system.

Therefore, on functions (i.e. rank (0,0) tensors),

(3.2) $\boxed{\mathcal{L}_V(f) = V^\lambda \partial_\lambda f = \text{the directional derivative}}$

rank (0,0)

For a rank (k,ℓ) tensor in coord system (4.1a)

$$\mathcal{L}_V T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_\ell} = \frac{\partial}{\partial x^1} T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_\ell}$$

e.g. for rank (1,0) vector W, $\mathcal{L}_V(W)$ has components

$$\mathcal{L}_V(W)^\mu = \frac{\partial}{\partial x^1} W^\mu = \partial_1 W^\mu$$

Compare to

$$[V,W]^\mu = V^\lambda \partial_\lambda W^\mu - W^\lambda \partial_\lambda V^\mu$$
$$= \frac{\partial}{\partial x^1} W^\mu - \phi$$
$$= \partial_1 W^\mu \; ! \; ☺$$

(3.3) $\Rightarrow \boxed{\mathcal{L}_V(W) = [V,W] \text{ for vectors}}$

⊛ For an arbitrary rank (k,ℓ) tensor, (see Carroll § App. B)

(4.1)
$$\boxed{\begin{aligned}\mathcal{L}_V (T)^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_\ell} = {} & V^\sigma \nabla_\sigma T^{\mu_1\mu_2\cdots\mu_k}{}_{\nu_1\nu_2\cdots\nu_\ell} \\ & - (\nabla_\lambda V^{\mu_1}) T^{\lambda\mu_2\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} \\ & - \cdots \\ & - (\nabla_\lambda V^{\mu_k}) T^{\mu_1\cdots\mu_{k-1}\lambda}{}_{\nu_1\cdots\nu_\ell} \\ & + (\nabla_{\nu_1} V^\lambda) T^{\mu_1\cdots\mu_k}{}_{\lambda\nu_2\cdots\nu_\ell} \\ & + \cdots \\ & + (\nabla_{\nu_\ell} V^\lambda) T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_{\ell-1}\lambda}\end{aligned}}$$

• Note especially the expression for the metric tensor:

(4.2)
$$\boxed{\mathcal{L}_V (g)_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu}$$

because $\nabla g = 0$ (metric-compatible connection)

(4.3) So if $\nabla_{(\mu} V_{\nu)} = 0$, metric unchanged along $\int$ curves of V.

Killing tensors

- Consider our friend the geodesic equation

(5.1) $$p^\lambda \nabla_\lambda p^\mu = 0$$

- Suppose we have a vector field K, cpts K^μ, for which Lie derivative of g is zero: $\nabla_{(\mu} K_{\nu)} = 0$. This means we have a <u>symmetry</u> of spacetime metric under pushing tensors along integral curves of K!

⊛ Consider quantity $K \cdot p$:

(5.2) $$\nabla_\mu (K_\lambda p^\lambda) = (\nabla_\mu K_\lambda) p^\lambda + K_\lambda (\nabla_\mu p^\lambda) = ¿\ 😐\ ?$$

Well,

(5.3) $$p^\mu \nabla_\mu (K_\lambda p^\lambda) = \underbrace{p^\mu p^\lambda \nabla_\mu K_\lambda}_{= p^\mu p^\lambda \nabla_{(\mu} K_{\lambda)}} + K_\lambda \underbrace{p^\mu (\nabla_\mu p^\lambda)}_{=0 \text{ by geodesic eqn.}}$$

(5.4) $$= 0 \text{ by } \boxed{\nabla_{(\mu} K_{\nu)} = 0} \Leftrightarrow \boxed{K \cdot p \text{ covariantly conserved}}$$

- Killing tensors obey

(5.5a) $$\boxed{\nabla_{(\mu} K_{\nu_1 \ldots \nu_\ell)} = 0}$$

(5.5b) $$\Leftrightarrow \quad \boxed{p^\mu \nabla_\mu (K_{\nu_1 \ldots \nu_\ell} p^{\nu_1} \ldots p^{\nu_\ell}) = 0}$$

Examples

(1) Minkowski space $\mathbb{R}^{1,3}$, with Cartesian coords, has
(6.1) $$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

(6.2) $\Rightarrow$ K.V.s $(1,0,0,0)$ $\Rightarrow$ energy p^0 conserved
& $(0,1,0,0)$
& $(0,0,1,0)$
& $(0,0,0,1)$ $\Rightarrow$ momentum p^i conserved $(i=1,2,3)$

$\boxed{\text{translations in all 4 } \{x^\mu\} \text{ directions}}$

(2) Consider flat spacetime with spherical polar coords:
(6.3) $$ds^2 = -c^2dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

- $\frac{\partial}{\partial\varphi}$ is a symmetry, manifestly. In Cartesian coords,

(6.4) $$\partial_\varphi = x\partial_y - y\partial_x \Rightarrow \text{K.V.s } (0,-y,x,0)$$
Similarly, & $(0,-z,0,x)$ & $(0,0,-z,y)$ $\boxed{\text{3 rotations}}$

(6.5) In polar coords, 2nd & 3rd K.V.s are $\begin{cases} -\cos\varphi\,\partial_\theta + \cot\theta\sin\varphi\,\partial_\varphi \\ \& \; -\sin\varphi\,\partial_\theta - \cot\theta\cos\varphi\,\partial_\varphi \end{cases}$

Note: no ∂_r components – these are ROTATIONS! ✓

[Try boosts for yourself...]