PLAN for today

tensor reminder

- Reminder on definition of tensors (2)
- &
- Index manipulation rules (1)

important tensor e.g.s

- 4-momentum p^μ (2)
- &
- EM field $F_{\mu\nu}$ (2)

shifting gears (toward GR)

- Constant acceleration (3)
- &
- "Twin Paradox"

Δt: 13:10–13:25 | 13:25–13:45 | 13:45–14:00

Last time, we defined a rank (k,ℓ) tensor $\mathbf{T}$ as ① multilinear map

(0.1) $$\boxed{\mathbf{T}: \underbrace{T_p^* \otimes \dots \otimes T_p^*}_{k \text{ factors}} \otimes \underbrace{T_p \otimes \dots \otimes T_p}_{\ell \text{ factors}} \longrightarrow \mathbb{R}}$$

i.e. a machine which takes
ℓ contravariant ("upstairs" components) vectors
& k covariant ("downstairs" components) vectors
and makes a real number;

and ② under coordinate transformations the tensor's components in the basis

(0.2) $$\boxed{\hat{e}_{(\mu_1)} \otimes \dots \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(\nu_1)} \otimes \dots \otimes \theta^{(\nu_\ell)}}$$

transform as

(0.3) $$\boxed{T^{\mu_1' \dots \mu_k'}{}_{\nu_1' \dots \nu_\ell'} = \Lambda^{\mu_1'}{}_{\lambda_1} \cdots \Lambda^{\mu_k'}{}_{\lambda_k} \Lambda^{\sigma_1}{}_{\nu_1'} \cdots \Lambda^{\sigma_\ell}{}_{\nu_\ell'} T^{\lambda_1 \dots \lambda_k}{}_{\sigma_1 \dots \sigma_\ell}}$$

(0.4) The transformation matrices are: $\boxed{\Lambda^{\mu'}{}_{\lambda} := \dfrac{\partial x^{\mu'}}{\partial x^{\lambda}}}$

(0.5) and their inverses are: $\boxed{\Lambda^{\sigma}{}_{\nu'} := \dfrac{\partial x^{\sigma}}{\partial x^{\nu'}}}$

For example:

(0.6) $\left\{\begin{array}{l}\text{Boost of rapidity } \zeta \\ \text{from } x \text{ to } x' \text{ coords} \\ \text{along } x^1\text{-axis}\end{array}\right\} \Rightarrow (\Lambda^{\mu'}{}_{\lambda})_{boo1} = \begin{bmatrix} \cosh\zeta & \sinh\zeta & 0 & 0 \\ \sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

["Λ"]

[This satisfies the $SO(1,3)$ requirement $\Lambda^T \eta \Lambda = \eta$ because $\cosh^2\zeta - \sinh^2\zeta = 1$ for all ζ.]

(0.7) Inverse matrix ["Λ^{-1}"] is $(\Lambda^{\sigma}{}_{\nu'})_{boo1} = \begin{bmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Under coord change, contravariant vector transforms

(0.8) as $V^{\mu'} = \Lambda^{\mu'}{}_{\lambda} V^{\lambda}$ i.e. under boost along x^1 see

$$V^{0'} = \cosh\zeta\, V^0 + \sinh\zeta\, V^1 \; ;$$
$$V^{1'} = +\sinh\zeta\, V^0 + \cosh\zeta\, V^1 \; ;$$
$$V^{2'} = V^2 \; ; \quad V^{3'} = V^3 .$$

just like 4-vector x^μ

but for covariant vector ω we find under boo1

(0.9)
$$\omega_{0'} = \cosh\zeta\, \omega_0 - \sinh\zeta\, \omega_1 \; ;$$
$$\omega_{1'} = -\sinh\zeta\, \omega_0 + \cosh\zeta\, \omega_1 \; ;$$
$$\omega_{2'} = \omega_2 \; ; \quad \omega_{3'} = \omega_3 .$$

$\Leftrightarrow$ MINUS SIGNS MATTER! ☺

Manipulating Indices

⊛ In general, it matters both
(a) whether an index is up or down, and
(b) where the index sits along the left-right direction. !

e.g. a (2,1) tensor might be

(1.1) $$S := S^{\alpha\beta}{}_{\gamma}\, \hat{e}_{(\alpha)} \otimes \hat{e}_{(\beta)} \otimes \hat{\theta}^{(\gamma)}$$

or

(1.2) $$T := T_{\mu}{}^{\nu\lambda}\, \hat{\theta}^{(\mu)} \otimes \hat{e}_{(\nu)} \otimes \hat{e}_{(\lambda)}$$

or

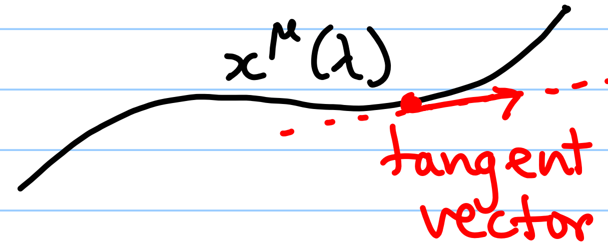
(1.3) $$U := U^{\sigma}{}_{\kappa}{}^{\tau}\, \hat{e}_{(\sigma)} \otimes \hat{\theta}^{(\kappa)} \otimes \hat{e}_{(\tau)}$$

⊛ We can relate upstairs & downstairs indices via the metric tensor, e.g. $V_\alpha = \eta_{\alpha\mu} V^\mu$,
BUT we cannot switch the left-right ordering of *any* pair of indices unless we know some symmetry or antisymmetry property for that particular tensor!

- Overall Goal of GR: $\boxed{\text{physical laws} = \text{nice tensor eqns}}$

Momentum 4-vector p^μ

Consider a particle moving along a "world-line"



Map from $\underset{\lambda}{\mathbb{R}}$ to $\underset{x^\mu}{\mathbb{R}^{1,3}}$

valid for massive particle

If we use proper time τ for λ, then the tangent vector is

(2.1) $$\boxed{\frac{dx^\mu}{d\tau} := u^\mu}$$ ← 4-velocity.

The components u^μ make up a 4-vector because the proper time is defined according to

(2.2) $$\boxed{-c^2 d\tau^2 := \eta_{\mu\nu}\, dx^\mu dx^\nu}$$ (*) (dimensions of length)

& because x^μ is a 4-vector.

(2.3) Then the 4-momentum is $\boxed{p^\mu := m u^\mu}$ if $m^2 > 0$

(2.4) and so $$\boxed{\{p^\mu\} = \left\{\frac{E}{c}, \vec{p}\right\}}.$$ ← (works for any m^2)

- Even for massless particles, p^μ still makes sense but is not m times u^μ!
- Note that u^μ is normalized such that $u^\mu u_\mu = \eta_{\mu\nu} u^\mu u^\nu = \eta_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -c^2$ by (2.2)

so we obtain

(3.1) $$\boxed{p^\mu p_\mu = -m^2c^2}$$

Works for $m^2 > 0$ and $m^2 = 0$.

i.e.

(3.2) $$\boxed{E^2 - |\vec{p}|^2 c^2 = (mc^2)^2}$$

transforms under Λ — same in ALL reference frames. (so is spin s of particle!)

(3.3) $\underline{m^2 > 0}$ In the particle's rest frame, $\{p^\mu\}_{rest} = mc\{1, \vec{0}\}$ and in another frame

(3.4) we get $\{p^\mu\} = mc\{\cosh\zeta, \hat{v}\sinh\zeta\}$

unit vector in direction of velocity $\vec{v}$ relating our 2 frames

(3.5) $\underline{m^2 = 0}$ $\{p^\mu\} = p\{1, \hat{v}\}$

$\boxed{\text{Electromagnetic Field } F_{\mu\nu}}$

- The electric ($\vec{E}$) and magnetic ($\vec{B}$) fields can be combined into a rank-two antisymmetric tensor F, whose components are $F_{\mu\nu}$.

- This is plausible at a basic level, because in $d=3+1$, a 4×4 antisymmetric tensor has
$$\frac{1}{2!}(4)(4-1) = \frac{1}{2}(4)(3) = 6 \text{ component} -$$
3 for $\vec{E}$ and 3 for $\vec{B}$.

⊛ We define a rank $(0,2)$ antisymmetric tensor F and write

(4.1) (4.2) $i \in \{1,2,3\}$ $\boxed{F_{0i} := -E_i}$ & $\boxed{F_{ij} = \sum_k \epsilon_{ijk} B_k}$

where

(4.3)
$$\hat{\epsilon}_{ijk} = \begin{cases} +1, & (ijk) \text{ even perm. of } (123) \\ -1, & (ijk) \text{ odd perm. of } (123) \\ 0, & \text{otherwise} \end{cases}$$

(4.4) e.g. $F_{01} = -E_1 \Rightarrow F_{10} = +E_1$

&

(4.5) e.g. $F_{12} = B_3$ $= B^3$ since spatial metric is $\mathbb{1}_3$.

(EM field, cont'd ...)

(5.1) • As a matrix,

$$(F_{\mu\nu}) = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ +E_1 & 0 & +B_3 & -B_2 \\ +E_2 & -B_3 & 0 & +B_1 \\ +E_3 & +B_2 & -B_1 & 0 \end{bmatrix}$$

(5.2) ⊛ Make a 4-vector from ρ and $\vec{J}$

$$\boxed{\partial_\mu J^\mu = 0}$$ current conservation

Then the Maxwell eqns become

(working out details on HW1)

(5.3)
$$\boxed{\partial_\nu F^{\nu\mu} = J^\mu}$$

and the other eqns not involving ρ & $\vec{J}$ are

(5.4)
$$\boxed{\partial_{[\mu} F_{\nu\lambda]} = 0}$$

[...] means antisym combo

→ Lorentz transformations are SO easy for $F_{\mu\nu}$!

(Working out how $\vec{E}$ & $\vec{B}$ transform is on HW1 too)

CONSTANT ACCELERATION

The "Twin Paradox" setup

- Einstein is famous for quite a few things. Possibly his most useful invention for physicists was his idea of the *thought experiment*. (= cheaper than a real one...)

- Let's imagine Bart, a homebody who stays on earth, and his astronaut twin who rockets off into outer space for awhile.

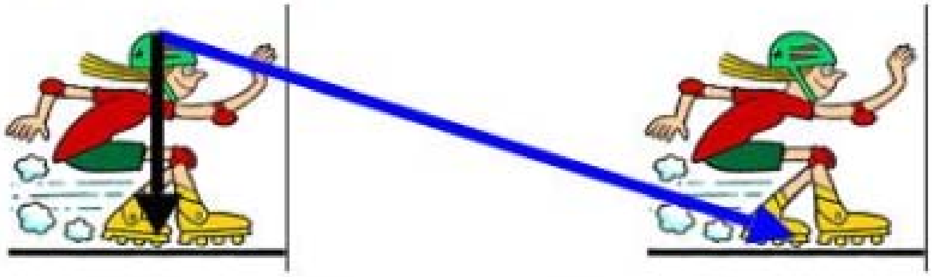
* Question: do they age the same?

- Old relativity would say yes - everyone keeps same time.
- Einstein said: actually, moving clocks run *slow*! And if one gets older quicker - who is it?

Time dilation

- *Why* did Einstein claim that moving clocks run slow?
- He figured it out by realizing that sending light-pulses between different places in space is the only sensible way to compare (and hence synchronize) different clocks.

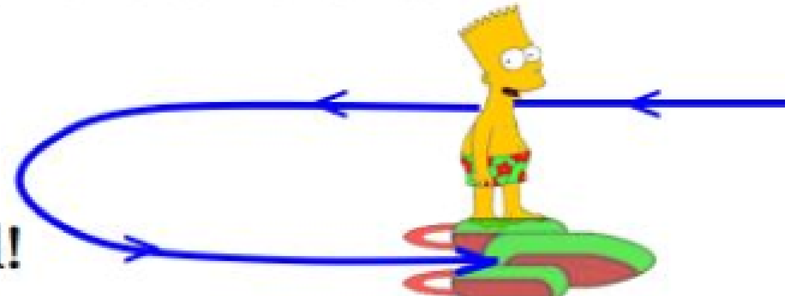
- Suppose you flash light from your head to your left toe, while you go by me on rollerblades. Then what do I see?



- Longer lightpath, same "c" $\Rightarrow$ I see your clock run slower!
- (Symmetrical: you also see mine run slower.)

Role of acceleration

- According to Einstein's special relativity, *each* Bart sees the other twin's clock running slower than theirs.



- Now, astronaut-Bart actually has to *accelerate*, to turn around!
- We can figure out time dilation for constant acceleration (by using calculus).

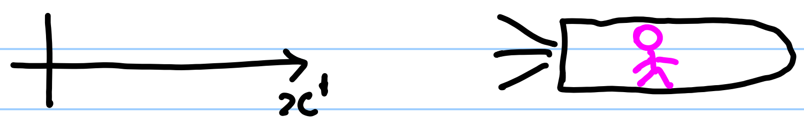
  Intuitively, *acceleration adds to the trip's time dilation*. The slower the acceleration, the smaller the extra effect.

- The acceleration of astronaut-Bart also breaks the symmetry between the twins. It gives us a reason to trust Einstein's theory, which says that <u>homebody-Bart always does age faster than astronaut-Bart</u>. ⇒ No paradox!

The constant-acceleration astronaut

When I was an undergrad, a prof. introduced the idea of the twin paradox – could the space-traveller twin really live longer by travelling at relativistic speeds? – but never equipped me with the technology to answer this question (!!). Here is how you can solve that.

- Consider a spaceship moving along the x direction in our stationary ("laboratory") frame.



constant acceleration g

The astronaut has a watch and measures proper time τ.

▷ How do we cope with the fact that this is not an inertial frame, i.e. not moving @ constant velocity?

* Approximate, at each instant of time, by an inertial frame of reference at that speed v.

(*) Mathematically, it is the rapidity ζ which is additive. i.e. under 2 successive Lorentz transformations it is ζ that adds, rather than velocities. ☺

- Infinitesimal addition to rapidity $d\zeta$ given in terms of acceleration g and proper time τ, because of our successive approximations by locally inertial frames

- UNITS? $\frac{v}{c} = \tanh\zeta$ i.e. ζ is dimensionless.

acceleration: $[a] = \frac{L}{T^2}$

proper time $[\tau] = T$

so $[a\tau] = \frac{L}{T} = [c]$

(8.1) Then $\boxed{d\zeta = \frac{g}{c}\,d\tau}$

- Each infinitesimal tick in astronaut proper time leads to an increase in rapidity measured in our/lab frame. Since g is constant, for our set-up, we get

(8.2) $\boxed{\zeta = \frac{g}{c}\,\tau}$

- Now we'd like to compute the distance in lab frame moved during lab time dt. This is simply given by the speed

(9.1) $$dx = v\,dt = c\tanh\zeta\,dt$$

- Next, to convert to astronaut time we use (4.2), and the time dilation formula

(9.2) $$dt = \frac{d\tau}{\sqrt{1-v^2/c^2}} = \cosh\zeta\,d\tau$$

i.e.

$$dx = c\tanh\zeta\,\cosh\zeta\,d\tau$$

(9.3) $$\Rightarrow\quad dx = c\sinh\zeta\,d\tau$$

- This we can integrate! Assuming that $x(\tau=0)=0$, we have

$$x(\tau) = c\int d\tau\,\sinh\left(\frac{g\tau}{c}\right)$$

(9.4) $$\Rightarrow\quad \boxed{x(\tau) = \frac{c^2}{g}\left[\cosh\left(\frac{g\tau}{c}\right)-1\right]}$$

- Also assuming $t(\tau=0)=0$ (initial condition) (8.2) $\Rightarrow$ $\boxed{t(\tau) = \frac{c}{g}\sinh\left(\frac{g\tau}{c}\right)}$ (9.5)