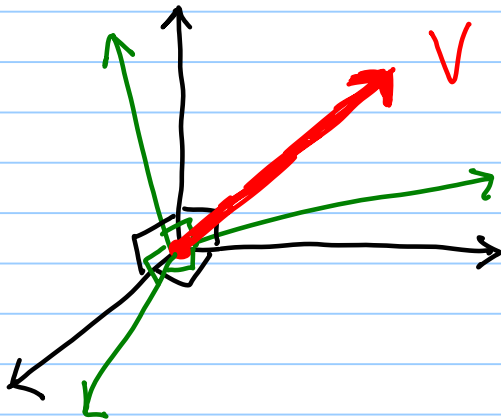


Vectors

①

When thinking about how vectors respond to coordinate changes, there are various ways to think about it.

We take the point of view (like in Carroll) that the vector stays fixed while the coordinate system changes under the relevant transformation. (⊕)



In the basis $\hat{e}_{(\mu)}$, V has components V^μ :

$$(1.1) \quad V = V^\mu \hat{e}_{(\mu)}.$$

Under a change of coordinates $x \rightarrow x'$, the components change as

$$(1.2) \quad V^{\mu'} = \Lambda^{\mu'}_{\nu} V^\nu \quad \text{where} \quad \Lambda^{\mu'}_{\nu} = \frac{\partial x^{\mu'}}{\partial x^\nu}$$

Since V is just a vector which sits there, V remains invariant. (⊕)

$$(1.3) \quad \text{So } \hat{e}_{(\mu')} = \Lambda^{\nu}_{\mu'} \hat{e}_{(\nu)} \quad \text{where } (\Lambda^{\nu}_{\mu'}) \text{ is the inverse matrix}$$

how basis vectors transform.

(2)

Dual Vectors

As in linear algebra, we can define a dual basis $\hat{\Theta}^{(\mu)}$ by

$$(2.1) \quad \hat{\Theta}^{(\mu)} \hat{E}^{(\nu)} = \delta^{\mu\nu} = \begin{cases} +1, & \mu = \nu \\ 0, & \text{otherwise} \end{cases}$$

"Kronecker delta" \uparrow

A dual vector ω can be written in components

$$(2.2) \quad \omega = \omega_{\mu} \hat{\Theta}^{(\mu)}$$

\nwarrow downstairs index

These are also called "one-forms" or "covariant vectors".

Under coord transformations, components transform:

$$(2.3) \quad \omega_{\mu'} = \Lambda^{\nu}_{\mu'} \omega_{\nu}$$

The inverse matrix \nearrow satisfies

$$\Lambda^{\nu}_{\mu'} \Lambda^{\mu'}_{\lambda} = \delta^{\nu}_{\lambda} \quad \text{and}$$

$$\Lambda^{\mu'}_{\nu} \Lambda^{\nu}_{\lambda'} = \delta^{\mu'}_{\lambda'} \quad (\text{i.e. } \Lambda \Lambda^{-1} = \mathbb{1} \text{ \& } \Lambda^{-1} \Lambda = \mathbb{1})$$

Math:

Dual vectors live in the cotangent space T_p^* which is a vector space.

ω 's are bilinear maps $\omega: T_p \rightarrow \mathbb{R}$:

$$(a\omega_1 + b\omega_2)(V) = a\omega_1(V) + b\omega_2(V)$$

$$\omega(aV_1 + bV_2) = a\omega(V_1) + b\omega(V_2) \quad .$$

Vectors & dual vectors, cont'd

⑧

A coordinate basis is when $\hat{\theta}^{(\mu)} = dx^\mu$;
then for some function ϕ

$$(3.1) \quad d\phi = \frac{\partial \phi}{\partial x^\mu} dx^\mu = \frac{\partial \phi}{\partial x^\mu} \hat{\theta}^{(\mu)}$$

Then we can call the components $\frac{\partial \phi}{\partial x^\mu}$
the "gradient" dual vector

$$(3.2) \quad \text{Notation:} \quad \frac{\partial}{\partial x^\mu} \equiv \partial_\mu$$

and

$$(3.3) \quad \partial_\mu \phi \equiv \phi_{,\mu}$$

← comma sign

For any vector V and dual vector ω

$$(3.4) \quad \text{the scalar } \omega(V) = V(\omega) = V^\mu \omega_\mu .$$

Notice that a vector cannot be "turned into"
a scalar without the help of either
(a) something with a downstairs index (eg. ω)
or (b) a metric tensor.

* The object with components $\eta_{\mu\nu} V^\mu$
is a dual vector(!)

Now suppose we consider something with (say)
components $\omega_\mu V^\lambda$. These indices are NOT
contracted (only repeated indices are summed over).
So ... is it δ^μ_λ ? Nope.

It is something new - a 4x4 matrix ...

TENSORS

Think of tensors as multi-legged generalizations of vectors and dual vectors.

Mathematically, a tensor T is a multilinear map

(4.1) $T: \underbrace{T_p^* \otimes \dots \otimes T_p^*}_{k \text{ times}} \otimes \underbrace{T_p \otimes \dots \otimes T_p}_{l \text{ times}} \rightarrow \mathbb{R}$

← tensor product

It is a 'machine' with $\begin{cases} k \text{ slots for vectors} \\ l \text{ slots for dual vectors} \end{cases}$ that spits out a scalar.

T has "type" or "rank" (k, l) .

Under coord transformations (viewpoint changes)

(4.2) $T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} = \Lambda^{\mu_1}_{\lambda_1} \dots \Lambda^{\mu_k}_{\lambda_k} \Lambda^{\sigma_1}_{\nu_1} \dots \Lambda^{\sigma_l}_{\nu_l} T^{\lambda_1 \dots \lambda_k}_{\sigma_1 \dots \sigma_l}$

This is like lots of matrix multiplication 😊

e.g. for rank $(0, 1)$ tensor T we have $T_{\nu'} = \Lambda^{\sigma}_{\nu'} T_{\sigma}$

i.e. $T_{\nu'} = T_{\sigma} \Lambda^{\sigma}_{\nu'}$ ←

[As a row vector with cpts $\{T_{\nu'}\}$ this eqn says $(T') = (T)(\Lambda^{-1})$ where $\{\Lambda^{\mu}_{\nu}\}$ is " (Λ) "]

You can express T in terms of basis tensors

$\hat{e}^{(\mu_1)} \otimes \dots \otimes \hat{e}^{(\mu_k)} \otimes \hat{\delta}^{(\nu_1)} \otimes \dots \otimes \hat{\delta}^{(\nu_l)}$

T 's components in this basis are just $T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l}$.

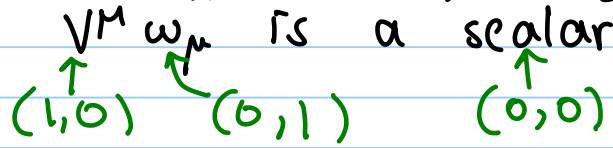
Vectors are $(1, 0)$ tensors.
Dual vectors are $(0, 1)$ tensors.

Now... what use are these things? Let's set some notational conventions to help.

Manipulating tensors

(see also Prof. Luke's QFT-1 notes, pp 9-12...)

- A way to reduce the number of legs on a tensor is to "contract" indices. You already know that $V^\mu w_\mu$ is a scalar where $\{V^\mu\}$ are vector cpts and $\{w_\mu\}$ dual vector cpts.



For a rank (k, l) tensor T : we can sum over a repeated index to make a rank $(k-1, l-1)$ tensor if one index is upstairs and one downstairs:

e.g.

(5.1) $S^\mu{}_{\lambda\sigma} \equiv T^{\mu\nu}{}_{\lambda\nu\sigma}$

N.B.: The ordering of indices matters!

Just like it matters whether you talk about M or M^T for a matrix M you know from linear algebra.

(5.2) In general, $T^{\mu\nu\lambda\sigma} \neq T^{\nu\mu\lambda\sigma}$, etc.

- Indices can be raised or lowered using the metric tensor $\eta_{\mu\nu}$ or its inverse $\eta^{\lambda\sigma}$.

(5.3) $\eta_{\mu\nu} \eta^{\nu\lambda} = \delta_\mu^\lambda$

All happen to be symmetric tensors
e.g. $\eta_{\mu\nu} = \eta_{\nu\mu}$
(as a matrix, $\eta = \eta^T$.)

e.g.

(5.4) $T^\mu{}_{\lambda\sigma} = \eta^{\mu\nu} T_{\nu\lambda\sigma}$

(5.5) and $S_{\lambda\mu\nu} = \eta_{\mu\sigma} S_{\lambda}{}^\sigma{}_\nu$

ordering matters!

Trace! for rank (n, n) tensors define the scalar
(5.6) e.g. $\text{Tr}(T) = T^\mu{}_{\nu\mu}$ (contract all indices)

Symmetry and Antisymmetry

A tensor T is symmetric if, under exchange of indices, it is the same.

(6.1) e.g. $T_{\mu\nu} = T_{\nu\mu}$

(6.2) or $S_{\mu\nu\lambda} = S_{\lambda\mu\nu} = S_{\mu\lambda\nu} = S_{\lambda\nu\mu} = S_{\nu\mu\lambda} = S_{\nu\lambda\mu}$

(If it only has symmetry on k of its n indices it is not 'completely symmetric'.)

You can see that symmetry reduces the # of independent components!

Symmetrizing } a tensor on k of its indices
or anti-symmetrizing } can be done only if all k of them are all-upstairs or all-downstairs.

Then e.g.

(6.3) $T_{(\mu_1 \dots \mu_k)}^{\lambda \sigma} \equiv \frac{1}{k!} (T_{\mu_1 \dots \mu_k}^{\lambda \sigma} + T_{\mu_2 \mu_1 \dots \mu_k}^{\lambda \sigma} + \text{perms.})$

e.g.

(6.4) $S_{[\mu_1 \dots \mu_k]}^{\lambda \sigma} \equiv \frac{1}{k!} (S_{\mu_1 \dots \mu_k}^{\lambda \sigma} - S_{\mu_2 \mu_1 \dots \mu_k}^{\lambda \sigma} + \text{perms.})$
[alternating sign]

For a rank $(2,0)$ or $(0,2)$ tensor, it can always be written as the sum of a symmetric part and an antisymmetric part:

$$T_{\mu\nu} = S_{\mu\nu} + A_{\mu\nu}$$

because

$$4 \times 4 = \frac{1}{2!} (4)(4+1) + \frac{1}{2!} 4(4-1)$$

In general for higher-rank tensors this sym + antisym split cannot be done: there aren't enough independent components.