Norm via Minkowski metric

- We are used to Euclidean norm, but sweep that into the history dustbin.
- Consider the implications of our new Minkowski metric in d=1+3.

Let's see what it does if we do with 4-vectors with η what we usually do to find the norm of a 3-vector. With $\vec{x}$ we find $\vec{x}^T$ to get

$$\|\vec{x}\|^2 = \vec{x}^T \cdot \vec{x} = (x^1)^2 + (x^2)^2 + (x^3)^2$$

$$= \vec{x}^T \cdot \mathbb{1} \cdot \vec{x}$$

this "2" signifies the y coörd not the square of x (!)

Now let's try to mimic this :–

$$x^T \eta\, x = \begin{bmatrix} x^0 & x^1 & x^2 & x^3 \end{bmatrix} \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

$$= \begin{bmatrix} x^0 & x^1 & x^2 & x^3 \end{bmatrix} \begin{bmatrix} -x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

$$\boxed{x \cdot x = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2}$$

Dot product of two 4-vectors

- For vector x^μ

$$\boxed{x \cdot x = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2}$$

this is not positive definite (!).

In particular, the 4-vector has norm

$$x \cdot x = -(x^0)^2 + \|\vec{x}\|^2$$
$$= -c^2 t^2 + \|\vec{x}\|^2$$

A fancier way of writing this is to say

$$x \cdot x = \eta_{\mu\nu} x^\mu x^\nu$$

More generally, for 4-vectors like x^μ,

$$\boxed{V \cdot W = \eta_{\mu\nu} V^\mu W^\nu \equiv \eta(V, W)}$$

where $(\eta_{\mu\nu}) \equiv \mathrm{diag}(-1, +1, +1, +1)$

So this metric takes two vectors & makes a scalar.
like a '2-slot machine'.

Light-cone

Interval classification

We had the norm of the 4-vector

$$x \cdot x = \eta(x, x) = x^\mu \eta_{\mu\nu} x^\nu$$

$$= -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$$

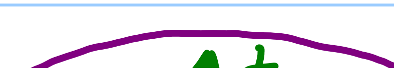
For a 4-vector Δx^μ denoting the space-time separation between 2 <u>events</u>, A & B,

$$\Delta x \cdot \Delta x = -c^2 \Delta t^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 .$$

Suppose we simplify by letting $\Delta y = 0$ & $\Delta z = 0$.
then $\Delta x \cdot \Delta x = -c^2 \Delta t^2 + \Delta x^2$.

$\Delta x \cdot \Delta x = 0 \iff$ A & B can be connected by a light ray.

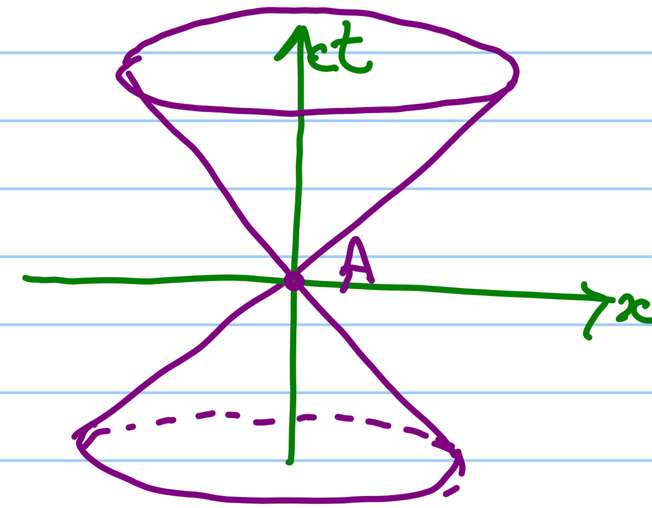
$$\left(\frac{\Delta x}{\Delta t} = \pm c \ ! \right)$$



Light-cone

$\overline{\Delta t}$;

If we put the origin at A,
all points B connectible to A
by a possible light ray
collectively form the
"Light cone" for A :



For $\Delta x \cdot \Delta x > 0$
a particle connecting A & B would have to go $> c$.
This is against the law. STOP e.g. $\Delta t = 0$, $\Delta x > 0$

For $\Delta x \cdot \Delta x < 0$, connecting can be done by real particles (not imaginary ones called "tachyons".)
e.g. $\Delta t > 0$, $\Delta x = 0$

Generally, a 4-vector V^μ is classified by its norm:-

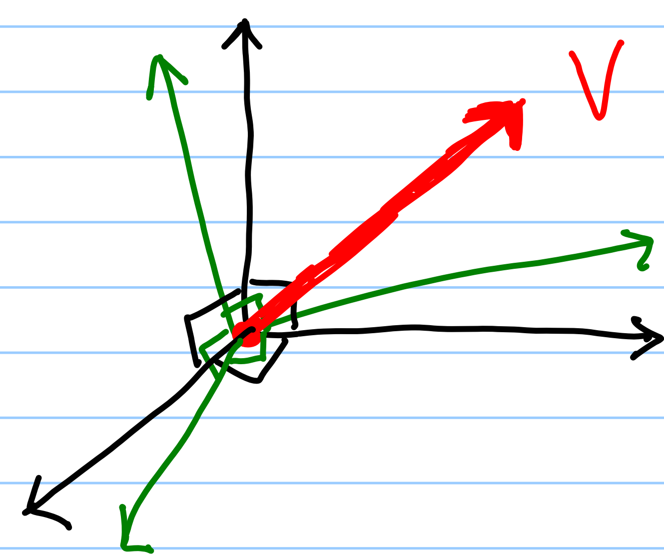
$$\eta_{\mu\nu} V^\mu V^\nu = \eta(V,V) = V \cdot V \begin{cases} = 0, & \text{lightlike} \\ < 0, & \text{timelike} \\ > 0, & \text{spacelike}. \end{cases}$$

Vectors (a.k.a. "contravariant vectors")

- Begin with familiar idea of 3-vector, broaden it to 1+3 dimensions.
- In our whole course, it matters whether indices μ are upstairs/downstairs!

When thinking about how vectors respond to coordinate changes, there are various ways to think about it.

We take the point of view (like in Carroll) that the vector stays fixed while the coordinate system changes under the relevant transformation. (ϕo)



What is a vector?

- Defined by behaviour under transformations to other frames of reference.
- Coordinate changes

also called a "contravariant vector"

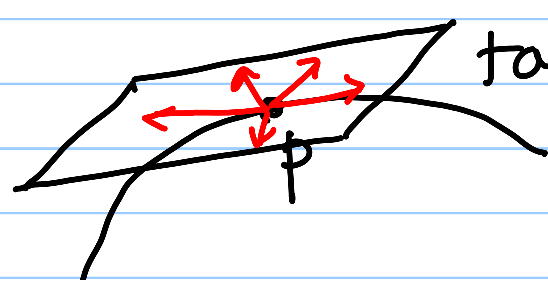
⑦

Definition of a (4-) vector

The physical characteristic of a 3-vector that matters for us is how it acts under rotations. (By contrast, a scalar — as distinct from a mountain climber, is oblivious to rotations.)

Similarly, what matters for 4-vectors, for us, is how they transform under rotations and boosts.

A vector is also associated to a point in spacetime - its base has to stick somewhere.



tangent plane at p

made up of vectors like these

→ "tangent space" T_p