GR 16 Sep 2005

Note Title 14/09/2004

- Philosophy: walk before running.
  ⇒ start by reviewing familiar material and repackaging it in new useful ways ☺

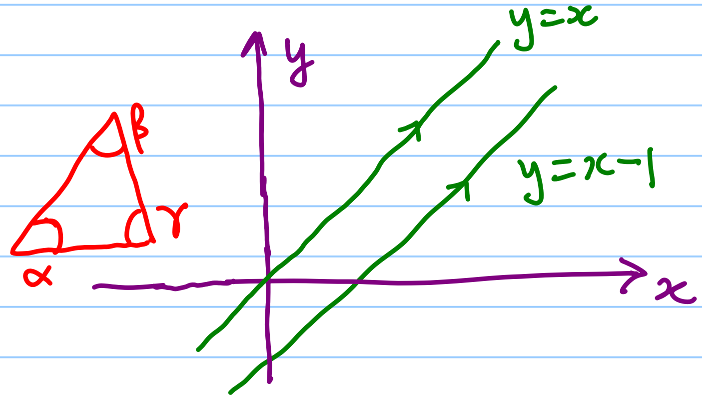
SPECIAL RELATIVITY

★ Today

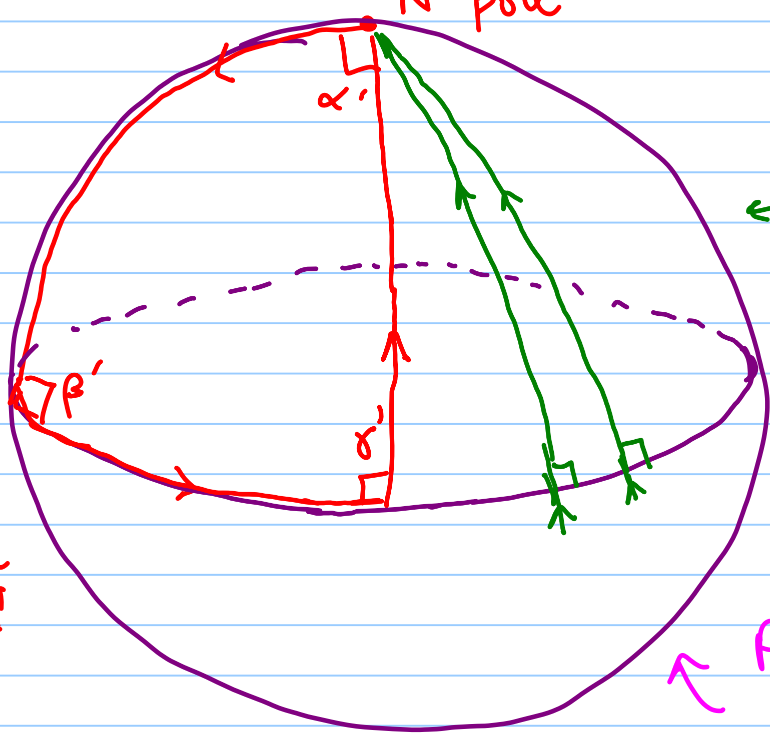
- §1.1 putting space and time together, the invariant interval
- §1.2 spacelike, lightlike and timelike intervals and the Minkowski metric
- §1.3 Lorentz transformations ("boosts") and how they differ from rotations
- §1.4 What is a vector?

Question: do parallel lines meet & do angles in a triangle add to 180° (π)?

First, consider the plane $\mathbb{R}^2$ i.e. flat two-plane:



Now let's look at (my bad drawing of) a 2-sphere "S^2"



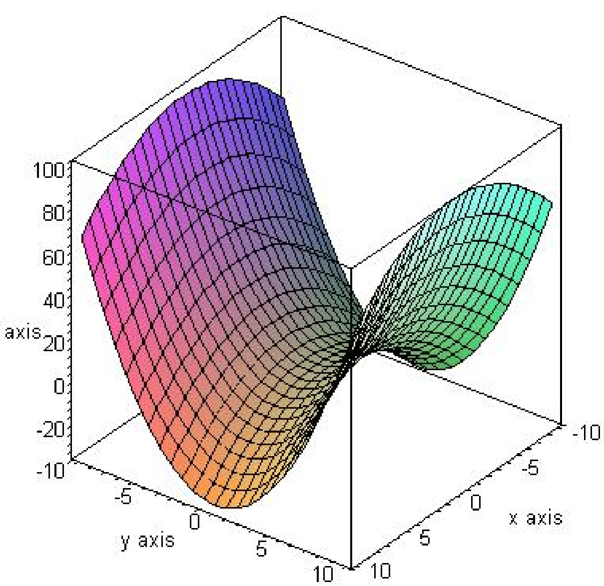
$\alpha' + \beta' + \gamma'$
$= \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2}$
$= \frac{3\pi}{2} > \pi$!

Each of these is obviously a right angle!

← these 2 lines start out ⊥ to equator as lines of longitude but meet @ North Pole!

Positively Curved Space

On the other hand, on the surface of a Pringle®,



angles of a triangle sum to LESS than π

&

parallel lines DIVERGE.

← Negatively Curved Space

Before curved spacetime comes $\boxed{\text{flat spacetime}}$

This week, we assume space-time is flat.

* Einstein taught us that simultaneity and length depend on your reference frame.

By contrast, Newton believed that clocks everywhere all keep the same time:

$$t' = t \quad \text{or} \quad \boxed{ct' = ct}$$

and the only spatial complication was to transform

$$x' = x - vt \quad \text{or} \quad \boxed{x' = x - \left(\frac{v}{c}\right)(ct)}$$

We now know that this was just the low-speed limit of the special relativity results

$$\boxed{\begin{aligned} ct' &= \gamma\left(ct - \frac{v}{c}x\right) \\ x' &= \gamma\left(x - \frac{v}{c}\cdot ct\right) \end{aligned}} \quad \text{and} \quad \text{where } \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$\left(\& \; \begin{aligned} y' &= y \\ z' &= z \end{aligned}\right)$

Unfortunately, the law of relativistic velocity addition is pretty ugly phrased in terms of v's & γ.

▷ Introduce rapidity ζ by $\boxed{\dfrac{v}{c} = \tanh \zeta}$

then $\gamma = \dfrac{1}{\sqrt{1 - \tanh^2 \zeta}} = \dfrac{\cosh \zeta}{\sqrt{\cosh^2 \zeta - \sinh^2 \zeta}} = \cosh \zeta$

so that

$$\begin{aligned} ct' &= (\cosh \zeta)(ct) - (\sinh \zeta)(x) \\ x' &= -(\sinh \zeta)(ct) + (\cosh \zeta)\, x \end{aligned}$$

(index "upstairs")

Let's write $(x^\mu) \equiv \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$, $\mu = 0, 1, 2, 3$ "Greek indices"

Four-vector

Then $$(x^\mu)' = \underbrace{\begin{bmatrix} (\cosh\zeta) & (-\sinh\zeta) & 0 & 0 \\ (-\sinh\zeta) & (\cosh\zeta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{This is a 4x4 matrix that looks awfully familiar ... like a rotation matrix ...} \\ \text{except for the fact that there are hyperbolic rather than trig functions.}}} (x^\mu)$$ "$(\Lambda^\mu{}_\nu)$"

So let's revisit old friends the rotations, now.

For the 3-vector $(x^i) = \vec{x}$ $\quad i = 1, 2, 3$ "Roman indices"
a rotation matrix we are used to takes
$$\vec{x} \to \vec{x}' = R\vec{x}$$

These are the matrices that are orthogonal
$$R^T R = \mathbb{1}$$
and have unit determinant ("SO(3)").

Let's rewrite this, kinda trivially:
$$R^T \mathbb{1}_3 R \overset{?}{=} \mathbb{1}_3$$

Now let's ask if Λ satisfies anything like this.
Find that
$$\Lambda^T \mathbb{1}_4 \Lambda \neq \mathbb{1}_4 \, . \quad ☹ \quad \leftarrow \text{Try it!}$$
But if we consider
$$\eta \equiv \text{diag}(-1, +1, +1, +1)$$
then
$$\boxed{\Lambda^T \eta \Lambda = \eta} \quad ☺$$

As a matrix, $$\boxed{(\eta_{\mu\nu}) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

Watch the index positioning...

it's very useful. [Good math --- = good def.]

Next time, we'll use $\eta_{\mu\nu}$ to
(a) take the norm of a vector
or the dot product of 2 vectors
(b) classify spacetime interval between 2 events
and hence understand flat-space causality
(c) and other things ☺

Here's a brief preview of (a):

Suppose we want the norm of x^μ.
We do it thusly:

$$x^\mu \eta_{\mu\nu} x^\nu = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$$

because $\eta_{0\nu} x^\nu = \eta_{00} x^0 + \eta_{01} x^1 + \eta_{02} x^2 + \eta_{03} x^3$

$$= \eta_{00} x^0 + (\text{zero})$$

$$= -x^0$$

Repeated Indices Are Always Summed Over

"Einstein Summation Convention"

while $\eta_{1\nu} x^\nu = \eta_{10} x^0 + \eta_{11} x^1 + \eta_{12} x^2 + \eta_{13} x^3$

$$= +x^1$$

and similarly

$$\eta_{2\nu} x^\nu = +x^2$$

with

$$\eta_{3\nu} x^\nu = +x^3$$

$\Rightarrow \quad x^\mu (\eta_{\mu\nu} x^\nu) = -(x^0)^2 + |\vec{x}|^2$ as claimed above.