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Light-Front Coordinates

For purposes of quantizing our string (later on), we will need to get familiar with a particular choice of (τ, σ) different from the static gauge studied so far.

Here is a lightning  review of these concepts.

(1.1) Define $x^{\pm} \equiv \frac{1}{\sqrt{2}}(x^0 \pm x^1)$

Then $-(dx^0)^2 + (dx^1)^2 = -2dx^+dx^-$

(1.2) so that the spacetime metric is $\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (\eta_{\mu\nu})$

Light-cone components of any vector or tensor can easily be found. Also,

(1.3) $a \cdot b = -a^+b^- - a^-b^+ + a^2b^2 + a^3b^3$

- In light-cone coords, let us inspect what physicists call the mass shell condition (it's one of two invariants of Poincaré: $p^\mu p_\mu$.)

$$p^\mu p_\mu = \eta_{\mu\nu} p^\mu p^\nu = -p^-p^+ - p^+p^- + \vec{p}_\perp^{\,2}$$

$$= -2p^-p^+ + \vec{p}_\perp^{\,2} = -m^2$$

$$\Rightarrow \quad p^-p^+ = \frac{1}{2}(m^2 + \vec{p}_\perp^{\,2})$$

(1.4) $$\boxed{-p_+ = p^- = \frac{1}{2p^+}(m^2 + \vec{p}_\perp^{\,2})}$$ "looks" non-relativistic (!)

(1.5) (where $p^{\pm} = \frac{1}{\sqrt{2}}\left(\frac{E}{c}, \pm p^1, 0, 0\right)$)

- Light-cone time is taken to be x^+.
  $\Rightarrow$ p_+ is the light-cone energy

String gauge choices: noncovariant gauges

Linear combination of position fields set to $\lambda\tau$:-

(2.1) $\boxed{n_\mu X^\mu(\tau,\sigma) = \lambda\tau}$ Breaks Lorentz invariance (via choice of particular n_μ).

The set of points x^μ satisfying $n_\mu x^\mu = \lambda\tau$ forms a hyperplane normal to n_μ.
$\Rightarrow$ the curve where the string worldsheet intersects this hyperplane normal to n_μ is the string at fixed τ.

What does this imply about energy-momentum? Static gauge! The momentum density was $\mathcal{P}^{\tau\mu}$, where energy is taken as "$\partial/\partial(\text{time})$". If instead we use the gauge (2.1), what we should demand is conservation not of $\mathcal{P}^{\tau\mu}$ but

(2.2) $$n_\mu \mathcal{P}^{\tau\mu} = \text{constant in } \tau \text{ \& } \sigma$$

This makes the string have constant energy density along its length (tension is constant in σ parametrization); the eqn (2.2) also ensures τ and σ are nicely orthogonal as we demanded earlier. Zwiebach shows that for an open string we can take $\sigma \in [0,\pi]$.

Equation of motion?

$$\frac{\partial}{\partial\tau}\mathcal{P}^{\tau\mu} + \frac{\partial}{\partial\sigma}\mathcal{P}^{\sigma\mu} = 0$$

$$\Rightarrow \frac{\partial}{\partial\tau}(n\cdot\mathcal{P}^\tau) = -\frac{\partial}{\partial\sigma}(n\cdot\mathcal{P}^\sigma)$$

(2.3) $$= 0 \Rightarrow (n\cdot\mathcal{P}^\sigma) \text{ independent of } \sigma.$$

For an open string, we required for free endpoints $\mathcal{P}^{\sigma\mu} = 0$ at the endpoints. But this implies $(n\cdot\mathcal{P}^\sigma) = 0$ at endpoints. Independence of σ then gives

(2.4) $$n\cdot\mathcal{P}^\sigma = 0 \text{ everywhere for open string (!)}$$

But for closed strings we have different BC's. Had

Momentum (Conserved): $\quad P_\mu = \int_\gamma (\mathcal{P}^\tau_\mu d\sigma - \mathcal{P}^\sigma_\mu d\tau)$

where γ winds once around cylindrical-topology worldsheet.
Now, choose purely spatial path along $\partial/\partial\sigma$:

(3.1) Let $\boxed{(n\cdot p)\,\sigma = \frac{2\pi}{\beta}\int_0^\sigma d\tilde{\sigma}\;\; n\cdot\mathcal{P}^\tau(\tau,\tilde{\sigma})}$

β to make $\begin{cases}\sigma\in[0,2\pi] & \text{for closed strings}\\ \sigma\in[0,\pi] & \text{for open strings}\end{cases}$

while

(3.2) $\boxed{n\cdot X(\tau,\sigma) = \beta\alpha'(n\cdot p)\,\tau}$ ← gauge choice with right dimensions, and tension, etc.

For the closed string, is $n\cdot\mathcal{P}^\sigma = 0$ like for open string?

$$n\cdot\mathcal{P}^\sigma = n_\mu\left\{\frac{-1}{2\pi\alpha'}\left[\frac{(\dot{X}\cdot X')\,\partial_\tau X^\mu - \dot{X}^2\partial_\sigma X^\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2(X')^2}}\right]\right\}$$

$$= \frac{-1}{2\pi\alpha'}\;\frac{(\dot{X}\cdot X')\,\partial_\tau(n\cdot X)}{\sqrt{(\dot{X}\cdot X')^2-\dot{X}^2(X')^2}} \qquad \text{using (3.2)}$$

In order to make this vanish, we would need

(3.3) $$\dot{X}\cdot X' = 0$$

on at least one point along the string; that would be enough to make it zero everywhere along σ by (2.3). Note that (3.3) is a relativistic dot product.

Is it zero somewhere?

Let $\{t^\mu\}$ the tangent vector to the worldsheet at a point P then $\{X'^\mu\}$ and $\{t^\mu\}$ generate T_P; we know they are not parallel because X'^μ is spacelike while t^μ is timelike. If they are orthogonal, i.e. $t^\mu X'_\mu = 0$, then t^μ is the tangent vector doing the job. Otherwise, define

$$v^\mu = t^\mu - \left(\frac{t\cdot X'}{X'\cdot X'}\right)X'^\mu \quad \text{which } \underline{\text{is}} \perp X'_\mu:\; v^\mu X'_\mu = 0.$$

Defining $\sigma=0$ to be the line given by

$$X^\mu(P) + \epsilon v^\mu$$

ensures that the tangent $\perp X'_\mu$ and, since the tangent vector $\propto \dot{X}^\mu$, have (3.3) satisfied. ⇒

σ can be any point on string.

$$\Rightarrow \boxed{n\cdot\mathcal{P}^\sigma = 0} \qquad \text{(open } \underline{\&} \text{ closed)}$$

$\frac{\partial}{\partial\sigma}$ symmetry unfixed!

(4.1) Having $\boxed{\dot{X}\cdot X' = 0}$ simplifies life.

$$\Rightarrow \mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\frac{(X')^2\dot{X}^\mu}{\sqrt{-(X')^2\dot{X}^2}} \quad \Rightarrow \quad n\cdot\mathcal{P}^\tau = \frac{1}{2\pi\alpha'}\frac{(X')^2(n\cdot\dot{X})}{\sqrt{-\dot{X}^2(X')^2}}$$

but $n\cdot X = \beta\alpha'(n\cdot p)\tau$ so $(n\cdot\dot{X}) = \beta\alpha'(n\cdot p)$

Also, since $(n\cdot p)\sigma = \frac{2\pi}{\beta}\int_0^\sigma d\tilde{\sigma}\,[\mathcal{P}^\tau(\tau,\tilde{\sigma})\cdot n]$

$$(n\cdot p) = \frac{2\pi}{\beta}\, n\cdot\mathcal{P}^\tau \quad \text{so}$$

$$n\cdot\mathcal{P}^\tau = \frac{\beta}{2\pi}(n\cdot p) = \frac{n\cdot\dot{X}}{2\pi\alpha'} = \frac{1}{2\pi\alpha'}\frac{(X')^2\, n\cdot\dot{X}}{\sqrt{-\dot{X}^2(X')^2}}$$

(4.2) so $\Rightarrow \frac{(X')^2}{\sqrt{-\dot{X}^2(X')^2}} = 1 \Rightarrow \boxed{\dot{X}^2 + (X')^2 = 0}$

Combining (4.1) & (4.2) gives $\boxed{(\dot{X}\pm X')^2 = 0}$

Momenta simplify:

(4.3) $$\boxed{\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu}$$

and

(4.4) $$\boxed{\mathcal{P}^{\sigma\mu} = \frac{-1}{2\pi\alpha'}X'^\mu}$$

(4.5) and so $\boxed{\ddot{X}^\mu - (X^\mu)'' = 0}$ equation of motion

The solution to the e-o-m (4.5) can be Fourier-expanded with the centre of mass motion split off:

(4.6a) $$\boxed{X^\mu_{\text{open}}(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\,\alpha_0^\mu\,\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\sigma)}$$

standing waves satisfy Neumann BC's @ endpoints.

where

(4.6b) $$\boxed{\alpha_0^\mu \equiv \sqrt{2\alpha'}\,p^\mu}$$

so that $(\dot{X}^\mu \pm X'^\mu) = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-in(\tau\pm\sigma)}$

Light-cone gauge (specific case)

We set $n_\mu = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0\right)$

so that $n \cdot X = X^+$ so $n \cdot p = p^+$ and

(5.1) $$\boxed{X^+(\tau,\sigma) = \beta\alpha' p^+ \tau}\,, \quad \boxed{p^+\sigma = \frac{2\pi}{\beta}\int_0^\sigma d\tilde{\sigma}\, \mathcal{P}^{\tau +}(\tau,\tilde{\sigma})}$$

Then requiring $(\dot{X} \pm X')^2 = 0$

(5.2) $$\Rightarrow \boxed{(\dot{X}^- \pm X^{-\prime}) = \frac{1}{\beta\alpha'}\frac{1}{2p^+}(\dot{X}^i \pm X^{i\prime})^2}$$

Using the general mode expansion (4.6a) and (5.1), see that

(5.3) $x_0^+ = 0$, $\alpha_n^+ = \alpha_n^- = 0$; $p^+ = \sqrt{2\alpha'}\,\alpha_0^+$

i.e. <u>in light-cone gauge, X^+ does not oscillate.</u>

Since we have (5.2), we can see that the $\ominus$ oscillators are going to be expressed in terms of quadratic combos of X^i oscillators.

In <u>detail, (see Zwiebach p.162)</u>, obtain [algebra]

(5.4) $$\boxed{\sqrt{2\alpha'}\,\alpha_n^- = \frac{1}{2p^+}\sum_{p\in\mathbb{Z}} \alpha^I_{n-p}\,\alpha^I_p}$$ (I is transverse)

So the α_n^- are <u>not</u> independent physical wiggles. It is the <u>X^I</u> that <u>have the "real" dynamics.</u> ☺

▷ Notice, for QFT students: when we set the gauge, (5.1), we got rid of <u>two</u> longitudinal degrees of freedom: not only did we lose X^+ oscillators via (5.1), the constraints knocked out X^- oscillators too. Quite generally, see that to knock out one longitudinal component of a vector, need to knock out a second. i.e. for A_μ in E&M also get only $(D-2)$ indep. d.o.f. (polarizations)

Virasoro modes

(6.1) Rename $\boxed{L_n^\perp \equiv \frac{1}{2} \sum_{p \in \mathbb{Z}} \alpha^I_{n-p} \cdot \alpha^I_p}$

(6.2) then $\boxed{2p^+ p^- = \frac{1}{\alpha'} L_0^\perp}$

(6.3) so that $\boxed{(\dot{X}^- \pm X^{-\prime}) = \frac{1}{p^+} \sum_{n \in \mathbb{Z}} L_n^\perp e^{-in(\tau \pm \sigma)}}$

We can also easily compute the mass in light-cone gauge:

$$M^2 = -p^\mu p_\mu = +p^+ p^- + p^- p^+ - p^I p^I$$
$$= \frac{1}{\alpha'} L_0^\perp - p^I p^I$$
$$= \frac{1}{\alpha'} \left[\frac{1}{2} \alpha_0^I \alpha_0^I + \sum_{n \in \mathbb{Z}^+} \alpha_n^{I*} \alpha_n^I \right] - p^I p^I$$

(6.4) $$\boxed{M^2 = \frac{1}{\alpha'} \sum_{n \in \mathbb{Z}^+} n\, a_n^{I*} a_n^I} \geq 0$$

Classically

where $\alpha_n^\mu = a_n^\mu \sqrt{n}$, $n \geq 1$.

Later on, we will quantize the string.

Each mode will contribute a zero-point energy, and we have to figure out the effect on the $(\text{mass})^2$ - once we know how to quantize!

First, we discuss the simpler cases of scalar, vector and tensor fields in light-cone gauge to get some warm-up practice with the canonical quantization procedure.

Then we will graduate to quantizing strings.