Example: computing Riemann using Vierbeins ☺

Note Title 30/09/2004

Previously we looked at a metric with

$$ds^2 = -dt^2 + t\,(d\vec{x}^2)$$

We can actually analyze a more generic class fairly easily, where

(1) $$\boxed{ds^2 = -dt^2 + a^2(t)\,d\vec{x}^2}$$

Carroll uses the variational technique to compute the Christoffel symbols which then can be used to find the curvature. Let's see an even easier method! (☺)

Let's write down the vierbeins, $e_\mu{}^{\hat a}$ such that

$$g_{\mu\nu} = \eta_{\hat a \hat b}\, e_\mu{}^{\hat a} e_\nu{}^{\hat b}$$

hatted indices are flat.

Here, we have $e_0{}^{\hat 0} = 1$ and $e_i{}^{\hat i} = a(t)$ ← (no sum on i)

Then

(2a) $$e^{\hat 0} = e_\mu{}^{\hat 0}\, dx^\mu = dt \quad \therefore \boxed{e^{\hat 0} = dt}$$

while

(2b) $$e^{\hat i} = e_\mu{}^{\hat i}\, dx^\mu = a(t)\, dx^{\hat i} \quad \therefore \boxed{e^{\hat i} = a\, dx^{\hat i}}$$

We require the torsion-free condition

$$de^{\hat a} + \omega^{\hat a}{}_{\hat b} \wedge e^{\hat b} = 0$$

Now,

$$de^{\hat 0} = d(dt) = 0 \quad \text{← (because } d^2 = 0 \text{ on a 0-form)}$$

while

$$de^{\hat i} = d\left(a(t)\, dx^{\hat i}\right) = \dot a\, dt \wedge dx^{\hat i} \qquad \text{where } \dot{} = \frac{d}{dt}$$

$$= (\dot a / a)\, e^{\hat 0} \wedge e^{\hat i}$$

So we need to find the $\omega^{\hat a}{}_{\hat b}$ satisfying

(3a) $$de^{\hat 0} + \omega^{\hat 0}{}_{\hat a} \wedge e^{\hat a} = 0$$

(3b) $$de^{\hat i} + \omega^{\hat i}{}_{\hat a} \wedge e^{\hat a} = 0$$

$\}$ $\hat a$ can be $\hat 0$ or $\hat j$

Now, the first eqn. here says

$$de^{\hat 0} = 0 = -\omega^{\hat 0}{}_{\hat a} \wedge e^{\hat a}$$

$$\Rightarrow \quad 0 = -\omega^{\hat 0}{}_{\hat 0} \wedge e^{\hat 0} - \omega^{\hat 0}{}_{\hat j} \wedge e^{\hat j}$$

$$= -\omega^{\hat 0}{}_{\hat 0} \wedge (dt) - \omega^{\hat 0}{}_{\hat j} \wedge (a\, dx^{\hat j})$$

(4a) $$\Rightarrow \quad \omega^{\hat 0}{}_{\hat 0} \propto dt \quad \text{or} \quad \omega^{\hat 0}{}_{\hat 0} = \alpha\, e^{\hat 0} \qquad \exists \alpha$$

(4b) $$\omega^{\hat 0}{}_{\hat j} \propto dx^{\hat j} \quad \text{or} \quad \omega^{\hat 0}{}_{\hat j} = \beta\, e^{\hat j} \qquad \exists \beta$$

How about (3.3b)?

$$de^{\hat{i}} + \omega^{\hat{i}}{}_{\hat{0}} \wedge e^{\hat{0}} + \omega^{\hat{i}}{}_{\hat{j}} \wedge e^{\hat{j}} = 0$$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right) e^{\hat{0}} \wedge e^{\hat{i}} = -\omega^{\hat{i}}{}_{\hat{0}} \wedge e^{\hat{0}} - \omega^{\hat{i}}{}_{\hat{j}} \wedge e^{\hat{j}}$$

$$= e^{\hat{0}} \wedge \omega^{\hat{i}}{}_{\hat{0}} - \omega^{\hat{i}}{}_{\hat{j}} \wedge e^{\hat{j}}$$

(5) We need the extra piece of information that $\boxed{\omega_{\hat{a}\hat{b}} = -\omega_{\hat{b}\hat{a}}}$ for the spin-connection

So $\omega^{\hat{i}}{}_{\hat{0}} = \omega^{\hat{0}}{}_{\hat{i}} = \beta e^{\hat{i}}$ works with (3.4b) & (4.1) provided that $\beta = (\dot{a}/a)$. It is also consistent to set $\alpha \equiv 0$ and $\omega^{\hat{i}}{}_{\hat{j}} = 0$.

$$\Rightarrow \boxed{\begin{aligned} \omega^{\hat{0}}{}_{\hat{0}} &= 0 \\ \omega^{\hat{0}}{}_{\hat{i}} &= (\dot{a}/a)\, e^{\hat{i}} \\ \omega^{\hat{i}}{}_{\hat{0}} &= (\dot{a}/a)\, e^{\hat{i}} \\ \omega^{\hat{i}}{}_{\hat{j}} &= 0 \end{aligned}} \qquad (6)$$

Now we can use these to compute the curvature 2-form. We have the structure equations $R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$, i.e.

$$R^{\hat{0}}{}_{\hat{0}} = d\omega^{\hat{0}}{}_{\hat{0}} + \omega^{\hat{0}}{}_{\hat{0}} \wedge \omega^{\hat{0}}{}_{\hat{0}} + \omega^{\hat{0}}{}_{\hat{j}} \wedge \omega^{\hat{j}}{}_{\hat{0}}$$

$$= 0 + 0 + (\dot{a}/a)^2 e^{\hat{j}} \wedge \delta_{\hat{i}\hat{j}} e^{\hat{i}} = 0$$

$$R^{\hat{0}}{}_{\hat{i}} = d\omega^{\hat{0}}{}_{\hat{i}} + \omega^{\hat{0}}{}_{\hat{j}} \wedge \omega^{\hat{j}}{}_{\hat{i}} + \omega^{\hat{0}}{}_{\hat{0}} \wedge \omega^{\hat{0}}{}_{\hat{i}}$$

$$= d\omega^{\hat{0}}{}_{\hat{i}} + 0$$

$$= d\left(\left(\frac{\dot{a}}{a}\right) e^{\hat{i}}\right) = d\left(\dot{a}\, dx^i\right) = \ddot{a}\, dt \wedge dx^i = \frac{\ddot{a}}{a} e^{\hat{0}} \wedge e^{\hat{i}}$$

$$R^{\hat{i}}{}_{\hat{0}} = d\omega^{\hat{i}}{}_{\hat{0}} + 0 + 0 = \frac{\ddot{a}}{a} e^{\hat{0}} \wedge e^{\hat{i}}$$

$$R^{\hat{i}}{}_{\hat{j}} = d\omega^{\hat{i}}{}_{\hat{j}} + \omega^{\hat{i}}{}_{\hat{0}} \wedge \omega^{\hat{0}}{}_{\hat{j}} + \omega^{\hat{i}}{}_{\hat{k}} \wedge \omega^{\hat{k}}{}_{\hat{j}}$$

$$= \left(\frac{\dot{a}}{a}\right) e^{\hat{i}} \wedge \left(\frac{\dot{a}}{a}\right) e^{\hat{j}} = \left(\frac{\dot{a}}{a}\right)^2 e^{\hat{i}} \wedge e^{\hat{j}}$$

(7) $$\Rightarrow \boxed{\begin{aligned} &R^{\hat{0}}{}_{\hat{i}\hat{0}\hat{j}} = (\ddot{a}/a)\, \delta_{\hat{i}\hat{j}} \; ; \quad R^{\hat{i}}{}_{\hat{0}\hat{0}\hat{j}} = (\ddot{a}/a)\, \delta^{\hat{i}}{}_{\hat{j}} \; ; \\ &R^{\hat{i}}{}_{\hat{j}\hat{k}\hat{l}} = (\dot{a}/a)^2\, \delta^{\hat{i}}{}_{\hat{k}}\, \delta_{\hat{j}\hat{l}} \end{aligned}}$$

(7.4) $$\Rightarrow \boxed{R_{\hat{0}\hat{0}} = -(d-1)\ddot{a}/a \; ; \quad R_{\hat{i}\hat{j}} = \delta_{\hat{i}\hat{j}}\left(\ddot{a}/a + (d-2)(\dot{a}/a)^2\right) \; ; \quad R_{\hat{i}\hat{0}} = 0}.$$

(7.5 $$\Rightarrow R = (d-1)\ddot{a}/a + (d-1)\ddot{a}/a + (d-1)(d-2)(\dot{a}/a)^2.$$

$$\boxed{R = 2(d-1)(\ddot{a}/a) + (d-1)(d-2)(\dot{a}/a)^2}$$

Can easily get $G_{\hat{\mu}\hat{\nu}}$ from these. (Try it! (exercise))