GR Wed 29 Sep 2004

Note Title 24/09/2004

Levi-Civita tensor

We previously defined $\tilde{\epsilon}$ as the permutation symbol;

(1.1) $$\tilde{\epsilon}_{\mu_0 \ldots \mu_3} = \begin{cases} +1, & (\mu_0 \ldots \mu_3) \text{ even perm. of } (0123) \\ -1, & \text{ odd} \\ 0, & \text{otherwise.} \end{cases}$$

E.g. $\tilde{\epsilon}_{0123} = +1$

It is completely antisymmetric (check for yourself!)

It's also very useful for representing determinants: for a 4x4 matrix M with components $M^{\mu}{}_{\nu}$

(1.2) $$\tilde{\epsilon}_{0123} (\det M) = M^{\alpha}{}_0 M^{\beta}{}_1 M^{\gamma}{}_2 M^{\delta}{}_3 \tilde{\epsilon}_{\alpha\beta\gamma\delta}$$

(by defn of det.)

Applying to $\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right)$, we have

(1.3) $$\tilde{\epsilon}_{\mu'_0 \ldots \mu'_{d-1}} = \underbrace{\left|\frac{\partial x'}{\partial x}\right|}_{\text{Jacobian}} \tilde{\epsilon}_{\nu_0 \ldots \nu_{d-1}} \Lambda^{\nu_0}{}_{\mu'_0} \ldots \Lambda^{\nu_{d-1}}{}_{\mu'_{d-1}}$$

; $\Lambda^{\nu}{}_{\mu'} = \frac{\partial x^{\nu}}{\partial x^{\mu'}}$ ($\neq$ constant)

Taking the det of $g_{\mu'\nu'} = \Lambda^{\lambda}{}_{\mu'} \Lambda^{\sigma}{}_{\nu'} g_{\lambda\sigma}$, see that

(1.4) $$\det(g_{..}(x')) = \left|\frac{\partial x'}{\partial x}\right|^{-2} \det(g_{..}(x))$$

Renaming $\sqrt{\det(g_{..}(x))} \equiv \sqrt{-g}$ ← signature $-+++$

(1.5) we have $\boxed{\sqrt{-g}\, \tilde{\epsilon}_{\mu \ldots \nu} = \epsilon_{\mu \ldots \nu} \text{ is a tensor}}$

Similarly,

(1.6) $\boxed{\left(\frac{1}{\sqrt{-g}}\right) \tilde{\epsilon}^{\mu \ldots \nu} = \epsilon^{\mu \ldots \nu} \text{ is a tensor.}}$ ← (not a mistake)

▷ Naive volume element $dx^0 dx^1 \ldots dx^n$ is not tensorial (!) Need a factor of $\sqrt{-g}$ to make it fly. ⇒ to integrate over spacetime, e.g. some scalar field $\phi(x^{\mu})$, we write

(1.7) $$\text{“}\int \phi\text{''} = \boxed{\int \sqrt{-g}\, d^4x\, \phi(x)}$$

This is a scalar i.e. sensible in GR.

Differential forms

▷ A p-form is just a completely antisymmetric (0,p) tensor. These are incredibly cool geometric objects that are useful for describing a limited class of physically interesting objects including electromagnetism. We can define several operations which are exclusive to forms. For a p-form in n dim's there are $\binom{n}{p} = \frac{n!}{(n-p)!\,p!}$ independent components.

(2.1) • $(A \wedge B)_{\mu_1 \dots \mu_{p+q}} = \frac{(p+q)!}{p!\,q!} A_{[\mu_1 \dots \mu_p} B_{\mu_{p+1} \dots \mu_{p+q}]}$ **Wedge product**

Example: $d = dx^\mu \partial_\mu$ (coord basis for (0,1) tensors)

$A = dx^\nu A_\nu$

then $(d \wedge A)_{\mu\nu} = 2\,\partial_{[\mu} A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$

or, more simply: $F = d \wedge A$

Note that $A \wedge B = (-1)^{pq} B \wedge A \neq B \wedge A$ in general!

(2.2) • $(dB)_{\mu_1 \dots \mu_{p+1}} = (p+1)\,\partial_{[\mu_1} B_{\mu_2 \dots \mu_{p+1}]}$ **Exterior derivative**

e.g. ϕ a function (0-form)

$d\phi = dx^\mu \partial_\mu \phi$

The amazing thing about d, the exterior derivative is that it DOES produce a tensor. The reason is that the failure of partial derivative to give a tensor was proportional to $\frac{\partial}{\partial x^\lambda}\left(\frac{\partial x^{\mu'}}{\partial x^\nu}\right) = \partial_\lambda \partial_\nu x^{\mu'}$ which is symmetric in $(\lambda\nu)$.

This cute fact is restricted only to p-forms, though.

(2.3) • Interesting property: $d^2 = 0$ on any p-form.

(e.g. for E&M, when $F = d \wedge A$ if we take

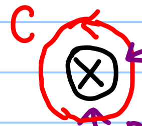
$A_2 = A_1 + d\lambda$, then $F_2 = d \wedge (A_1 + d\lambda) = d \wedge A_1 + d \wedge d \wedge \lambda = F_1$

so the gauge invariance in E&M is easy to see in the language of differential forms.)

BUT!

If $dG = 0$, this does not imply that G can be written as d of something, unless your spacetime manifold is topologically trivial. A form G such that $d \wedge G = 0$ is called closed; forms satisfying $G = d \wedge \beta \; \exists \beta$ are called exact.

A famous example in physics is called the Aharonov–Bohm effect.



Outside cylinder, $F_{\mu\nu}=0$ $(\vec{E}=\vec{0},\ \vec{B}=\vec{0}.) \Rightarrow F=0$

B-field into page

But A is not "pure gauge": it cannot be written as $d\beta$ $\nexists\beta$

We know A is not trivial because $\oint_C A \neq 0$

(3.1) • Stokes's Theorem is simply $\boxed{\int_M d\wedge\omega = \int_{\partial M} \omega}$, generally.

This is amazingly cool.

For our AB-effect example, $\oint_C A = \int_{\text{surface}} F \neq 0 \because$ of $\vec{B}$

▷ p-forms & $(n-p)$-forms have same # cpts. $\Rightarrow$ define

• $(*A)_{\mu_1\cdots\mu_{n-p}} = \frac{1}{p!}\,\epsilon^{\nu_1\cdots\nu_p}{}_{\mu_1\cdots\mu_{n-p}}\,A_{\nu_1\cdots\nu_p}$ Hodge dual

Then (check) $*(*A) = (-1)^{1+p(n-p)}$

one $\ominus$ sign in $(\eta_{\mu\nu})$

The amazing thing is how amazing Maxwell's eqns look in differential form notation.

The current J is a 1-form; then $*J$ is a $(n-1)$-form in n dimensions of space-time. Consider also dA, a 2-form. Then $*dA = *F$ is an $(n-2)$-form. Taking another d of this, we get an $(n-1)$-form. In fact, Maxwell's eqns are simply

$$\boxed{*d*F = J}\ ;$$

the requirement that $dF=0$ is an immediate consequence of defining $\boxed{F = d\wedge A}$

This is also called the Bianchi identity.

Gauge invariance is the statement that F is invariant under $A \to A + d\wedge\lambda$ (for arbitrary λ)

Covariant Derivative

Remember when we tried to make a tensor in curved space out of ∂(tensor) but failed, $\because \partial_\lambda (\Lambda^{\mu'}{}_{\nu}) \neq 0$ ☹
Let's fix this by inventing a new type of derivative which *does* make a new tensor out of an old one.

We define the covariant derivative ∇ to satisfy
(1) linearity : $\nabla(T+S) = \nabla T + \nabla S$
(2) Leibnitz rule : $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$
[like $\partial_\mu (ab) = (\partial_\mu a)\cdot b + a \cdot (\partial_\mu b)$ ...]

[not pronounced "Christ-awful"! ☺]

We define the Christoffel connection Γ by (in components)

(4.1) $$\boxed{\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})}$$ (σ is summed over)

and the covariant derivative on vectors by (again, in components)

(4.2) $$\boxed{\nabla_\mu V^\lambda = \partial_\mu V^\lambda + \Gamma^\lambda{}_{\mu\nu} V^\nu}$$

The important feature of this Γ thingy is that $(\nabla_\mu V^\nu)$ transforms properly i.e. as a bona fide (1,1) tensor!

N.B. (*) The proper derivation is quite boring and lengthy; I do not believe it wise to grind through it here in-class. It is much better to figure it out on your own time — you will appreciate it more this way (although it does take some "sweat" – but so do most good things in life! ☺)
See e.g. pp 97–98 of Carroll.

For a one-form: in components

(4.3) $$\boxed{\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda{}_{\mu\nu} \omega_\lambda}$$

(N.B: Neither term on the RHS here or in (4.1) is a bona fide tensor.)

It is also straightforward (but boring) to work out the rule for a general (k,l) tensor:–

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(5.1)
$$\boxed{\begin{aligned}\nabla_\sigma T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_\ell} = {} & \partial_\sigma T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_\ell} \\ & + \Gamma^{\mu_1}{}_{\sigma\lambda} T^{\lambda\mu_2\ldots\mu_k}{}_{\nu_1\ldots\nu_\ell} + \Gamma^{\mu_2}{}_{\sigma\lambda} T^{\mu_1\lambda\mu_3\ldots\mu_k}{}_{\nu_1\ldots\nu_\ell} \\ & + \ldots \\ & - \Gamma^{\lambda}{}_{\sigma\nu_1} T^{\mu_1\ldots\mu_k}{}_{\lambda\nu_2\ldots\nu_k} - \Gamma^{\lambda}{}_{\sigma\nu_2} T^{\mu_1\ldots\mu_k}{}_{\nu_1\lambda\nu_3\ldots\nu_\ell} + \ldots\end{aligned}}$$

This definition of ∇ has the other nice properties:

(3) commutes with contraction, i.e.

e.g. $\nabla_\mu (T^\lambda{}_{\lambda\rho}) = (\nabla T)_\mu{}^\lambda{}_{\lambda\rho}$ or $\nabla_\mu(\delta^\nu{}_\lambda) = 0$

(4) ∇ reduces to ∂ action on scalars

i.e. $\nabla_\mu \varphi = \partial_\mu \varphi$.

There is a discussion in Carroll (somewhat advanced) showing that (4.1) is actually the unique $\Gamma^\lambda{}_{\mu\nu}$ satisfying axioms (1)–(4) [on pp4-5 above] provided that

(5.2) (a) $\boxed{\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu}}$ ← "torsion-free"

and

(5.3) (b) $\boxed{\nabla_\sigma g^{\mu\nu} = 0}$ ← "metric-compatible"

There is a really simple way to look at these based on a story involving [vier]beins $e^a{}_\mu$:

(5.4) $g_{\mu\nu} = \eta_{ab}\, e^a{}_\mu e^b{}_\nu$

(5.5) Define $\omega^a{}_b = \omega^a{}_{b\mu} dx^\mu$ and $e^a = e^a{}_\mu dx^\mu$. Then

↖ "Spin connection"

(5.6) $T^a \equiv De^a = de^a + \omega^a{}_b \wedge e^b = 0$ is (5.3)

Then one may wonder what is → "Cartan structure eqns"

(5.7) $R^a{}_b \equiv D\omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$

This is called the curvature tensor and its components $R^a{}_{b[\mu\nu]}\, dx^\mu \wedge dx^\nu = R^a{}_b$ we will meet again soon.

It obeys the "Bianchi identity":

$DR^a{}_b = 0$

Stokes's Theorem

A special cool little identity is satisfied by this non-tensor the Christoffel connection:

(6.1) $$\boxed{\Gamma^\lambda{}_{\lambda\nu} = \frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}\right)}$$

$$\Rightarrow \quad \nabla_\mu V^\mu = \partial_\mu V^\mu + \frac{1}{\sqrt{-g}}(\partial_\mu\sqrt{-g})V^\mu$$

$$= \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,V^\mu\right)$$

and so

(6.2) $$\int_\Sigma (\nabla_\mu V^\mu)(\sqrt{-g}\, d^d x) = \int_{\partial\Sigma} n_\mu V^\mu \sqrt{-\gamma}\, d^{d-1}x$$

unit normal to $\partial\Sigma$ induced metric on $\partial\Sigma$
(See Appendix E for more.)

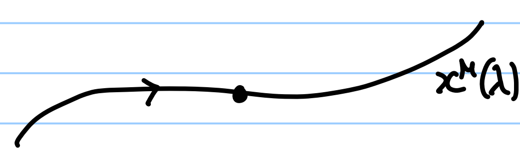
Parallel Transport

← really big, important idea!!

Introducing a covariant derivative was a great idea. ☺ It allows us to write tensor equations "wherever we go"; "all" we need to do is be sure to write D's (∇'s) rather than ∂'s. But... <u>what rate of change does D measure?</u>

A way to answer this question is to ask when ∇ of some tensor is zero, to get a handle on ∇.
For this, we actually have to specify what <u>path</u> along which we hope to compare tensors – because comparing tensors at two different points is, a priori, meaningless in GR: the spacetime metric differs!

So consider a path $x^\mu(\lambda)$ and define the <u>directional covariant derivative</u>



(6.3) $$\boxed{\frac{D}{d\lambda} \equiv \frac{dx^\mu}{d\lambda}\nabla_\mu}$$

(Carroll's funny notation)

This guy is defined φ only along the path $x^\mu(\lambda)$.

⊛ We say that a tensor T is <u>parallel-transported</u> along the path $x^\mu(\lambda)$ if

(7.1) $$\left(\frac{D}{d\lambda}T\right)^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} = \frac{dx^\sigma}{d\lambda}\nabla_\sigma T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_\ell} = 0$$

(a.k.a. "Equation of parallel transport")

- Notice that this is a nice proper tensor eqn because ∇T is a tensor and so is $\frac{dx}{d\lambda}$.
- Also, since $\nabla g = 0$, <u>parallel transport preserves the inner product of two tensors</u> ‡

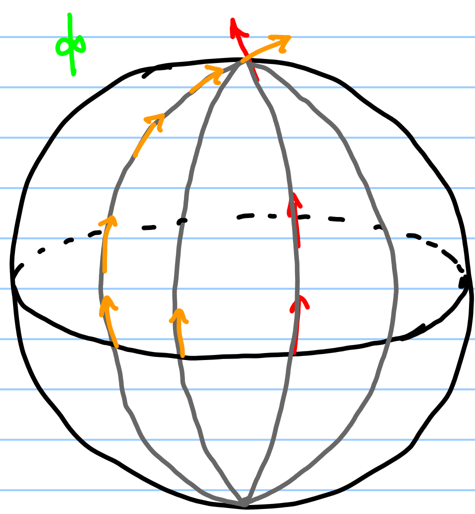
e.g.: for 2 vectors V & W,

$$\frac{D}{d\lambda}(g_{\mu\nu}V^\mu W^\nu) = \underbrace{\left(\frac{D}{d\lambda}g_{\mu\nu}\right)}_{=0}V^\mu W^\nu + g_{\mu\nu}\left[\underbrace{\left(\frac{D}{d\lambda}V^\mu\right)}_{=0}W^\nu + V^\mu\underbrace{\left(\frac{D}{d\lambda}W^\nu\right)}_{=0}\right]$$

$$= 0 \quad (\checkmark).$$

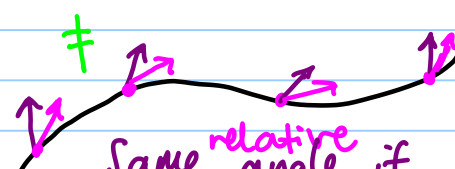
This is a crucial property of parallel transport.

φ



//el transport so depends on the path!

‡



Same relative angle if //el transported

⊛ A <u>geodesic</u> is a path that parallel-transports its own tangent vector:

(7.2) $$\boxed{\frac{D}{d\lambda}\left(\frac{dx^\mu}{d\lambda}\right) = 0 = \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\nu\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda}}$$

Nice ODE.

Remarks (not made in today's lecture):

We can also think about parallel transport, a notion that needs a ∇ (and hence a Γ), as follows.

When we take an ordinary derivative, we do it by, e.g. taking $\lim_{\Delta x^\mu \to 0} \frac{f(x^\mu + \Delta x^\mu) - f(x^\mu)}{\Delta x^\mu} = \frac{\partial f}{\partial x^\mu}$

In curved space, the result of this isn't a tensor. What we do to take the covariant derivative is:

(a) Take our tensor T at "$x^\mu + \Delta x^\mu$" where we measure Δx^μ along the path $x^\mu(\lambda)$: we take $x^\mu(\lambda + \Delta\lambda)$ as our "$x^\mu + \Delta x^\mu$" & find T there.

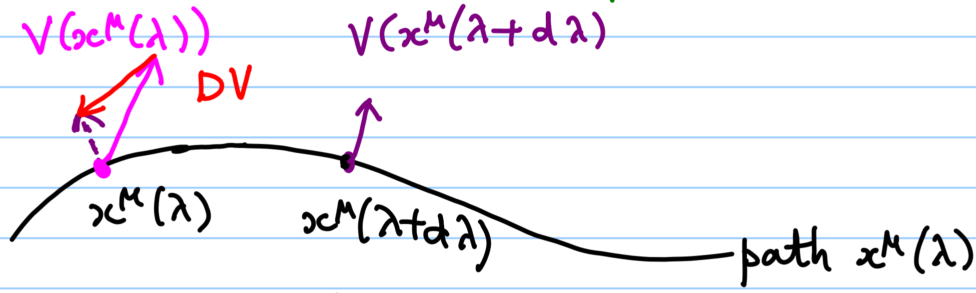
(b) We parallel-transport T back to $x^\mu(\lambda)$ along $x^\mu(\lambda)$.

(c) We compare the parallel-transported-back T to the actual T at $x^\mu(\lambda)$, and we divide by $\Delta\lambda$.

The result is $\frac{D}{d\lambda} T$.

E.g. for a 4-vector V with components V^μ:-

Pictorially:



& $\frac{DV}{d\lambda}$ is the directional covariant derivative along $x^\mu(\lambda)$

i.e. $\frac{DV^\nu}{d\lambda} = \frac{dx^\mu}{d\lambda} \nabla_\mu V^\nu$

is the "covariant" rate of change of V^ν w.r.t. λ at the spacetime point $x^\mu(\lambda)$.