Raychaudhuri's equation

Our Ricci tensor for spacetime is determined by $T_{\mu\nu}$ via Einstein's equations. What about the effect on matter of spacetime curvature?

(1.1a) (1.1b) Consider a congruence of geodesics $\begin{cases} u^\mu u_\mu = -1 \\ u^\lambda \nabla_\lambda u^\mu = 0 \end{cases}$

congruence: Set of curves in an open region of spacetime, such that every point in the region lies on precisely one curve.

Previously, we found that for the separation vector

(1.2) $$\frac{DV^\mu}{d\tau} = u^\nu \nabla_\nu V^\mu = B^\mu{}_\nu V^\nu$$

$$\boxed{B^\mu{}_\nu = \nabla_\nu u^\mu}$$

Consider space of vectors $\perp$ and $\parallel$ to u^μ.

- Use projector

(1.3) $$\boxed{P^\mu{}_\nu = \delta^\mu{}_\nu + u^\mu u_\nu}$$

to get the component of a tensor $\perp$ to u^μ.

- Claim: $B_{\mu\nu}$ is in the $\perp$ subspace.

Proof: $u^\mu B_{\mu\nu} = u^\mu \nabla_\nu u_\mu = 0 \because u^\mu u_\mu = -1$ so $u^\mu \nabla_\nu u_\mu = 0$

and $u^\nu B_{\mu\nu} = u^\nu \nabla_\nu u_\mu = 0$ by geodesic equation.

- $B_{\mu\nu}$ has trace, symmetric traceless, and antisymmetric components.

  $\boxed{\text{Note}}$: since $B \perp u$, use $P^\mu{}_\nu$ to take the trace! (cute mathematical trick / observation)

(1.4) Define $\theta \equiv P^{\mu\nu} B_{\mu\nu} = \boxed{\nabla_\mu u^\mu = \theta} \leftarrow$ "expansion" of congruence

Then define

$$\sigma_{\mu\nu} = B_{(\mu\nu)} - P_{\mu\nu} \cdot \frac{\theta}{3}$$

i.e.

(2.1) $$\boxed{\sigma_{\mu\nu} = \nabla_{(\nu} u_{\mu)} - (\delta^\mu{}_\nu + u^\mu u_\nu)\frac{1}{3}(\nabla_\lambda u^\lambda)}$$

there are 3 independent basis vectors in the ⊥ subspace (not 4!)

"<u>shear</u>" of the congruence

and thirdly define

(2.2) $$\boxed{\omega_{\mu\nu} = \nabla_{[\nu} u_{\mu]}}$$ "<u>rotation</u>" of the congruence

For evolution along the path,

$$\frac{D}{d\tau} = u^\sigma \nabla_\sigma$$

Acting on $B_{\mu\nu}$ we have

$$\begin{aligned}
\frac{D}{d\tau} B_{\mu\nu} &= u^\sigma \nabla_\sigma \nabla_\nu u_\mu \\
&= u^\sigma \left(\nabla_\nu \nabla_\sigma u_\mu + R^\lambda{}_{\mu\nu\sigma} u_\lambda \right) \\
&= \nabla_\nu (u^\sigma \nabla_\sigma u_\mu) - (\nabla_\nu u^\sigma)(\nabla_\sigma u_\mu) - R_{\mu\lambda\nu\sigma} u^\sigma u^\lambda
\end{aligned}$$

(2.3) $$= -B^\sigma{}_\nu B_{\mu\sigma} - R_{\mu\lambda\nu\sigma} u^\sigma u^\lambda$$

Tracing over this gives

(2.4) $$\boxed{\frac{D\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}u^\mu u^\nu}$$

This is called "<u>RAYCHAUDHURI'S EQUATION</u>"

Other parts (for $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$) are in Appendix F of Carroll on p.461.
(They don't get used nearly as often as (5.3).)

▷ How about physical consequences?

- Since $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ are both spatial tensors, (because $B_{\mu\nu} = \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{3} P_{\mu\nu}\Theta \perp u^\mu \leftarrow$ timelike)

(3.1) $$\omega_{\mu\nu}\omega^{\mu\nu} \geq 0 \quad \text{and} \quad \sigma_{\mu\nu}\sigma^{\mu\nu} \geq 0 .$$

- If u^μ is orthogonal to a family of hypersurfaces

(3.2) then $(u_{[\mu}\nabla_\nu u_{\rho]} = 0$ and) $\omega_{\mu\nu} \equiv 0$.

- Let's also consider

$$R_{\mu\nu}u^\mu u^\nu$$

Since $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N T_{\mu\nu} - \Lambda g_{\mu\nu}$,

(3.3) $$\therefore R\left(1 - \frac{D}{2}\right) = 8\pi G_N T - \Lambda D ,$$

$(D>2)$

$$R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}\frac{2}{(2-D)}\left(8\pi G_N T - \Lambda D\right) + 8\pi G_N T_{\mu\nu} - \Lambda g_{\mu\nu}$$

$$= \frac{1}{(2-D)}g_{\mu\nu}\, 8\pi G_N T - \frac{\Lambda D g_{\mu\nu}}{(2-D)} + 8\pi G_N T_{\mu\nu} - \Lambda g_{\mu\nu}$$

(3.4) $$= 8\pi G_N\left(T_{\mu\nu} - \frac{T g_{\mu\nu}}{(D-2)}\right) + g_{\mu\nu}\Lambda\frac{2}{(D-2)}$$

(3.5) $$\Rightarrow R_{\mu\nu}u^\mu u^\nu = 8\pi G_N\left[T_{\mu\nu}u^\mu u^\nu + \frac{T}{(D-2)}\right] - \frac{2}{(D-2)}\Lambda$$

This implies that

(3.6) if
- $\omega_{\mu\nu} = 0$
- $T_{\mu\nu}u^\mu u^\nu \geq -\frac{T}{(D-2)}u^\mu u_\nu$ $\leftarrow$ (Strong Energy Condition)
- $\Lambda \leq 0$

then $\frac{d\theta}{d\tau} \leq 0$

and the geodesics converge!

If $\Lambda > 0$ this helps push geodesics apart, which happens owing to (accelerated) expansion.

ENERGY CONDITIONS

(4.1) For a perfect fluid, $T_{\mu\nu}^{(p.f.)} = (\rho+p)u_\mu u_\nu + p g_{\mu\nu}$

Pressure is isotropic so $T_{\mu\nu}t^\mu t^\nu \geq 0 \ \forall$ timelike t^μ
if both $T_{\mu\nu}u^\mu u^\nu \geq 0$
and $T_{\mu\nu}\ell^\mu \ell^\nu \geq 0 \ \exists$ null vector ℓ^μ.

Regard
$$\begin{cases} T_{\mu\nu}u^\mu u^\nu = \rho \\ T_{\mu\nu}\ell^\mu\ell^\nu = (p+\rho)(u_\mu \ell^\mu)^2 \end{cases}$$

so we're just said $T_{\mu\nu}t^\mu t^\nu \geq 0 \Rightarrow \rho \geq 0$ & $(p+\rho) \geq 0$.

This is called the Weak Energy Condition.
Others include (& perfect fluid examples :-) :-

(4.2) • WEC : weak energy condition
$T_{\mu\nu}t^\mu t^\nu \geq 0$, $\forall t^\mu$ timelike vectors
$(\rho \geq 0$ and $p+\rho \geq 0)$

(4.3) • NEC: null energy condition
$T_{\mu\nu}\ell^\mu\ell^\nu \geq 0$ $\forall \ell^\mu$ null vectors
$(p+p \geq 0$; note that here ρ can be $<0)$

(4.4) • DEC: dominant energy condition
$T_{\mu\nu}t^\mu t^\nu \geq 0$ $\forall t^\mu$ timelike AND $T_{\mu\nu}T^\nu{}_\lambda t^\mu t^\lambda \geq 0$
i.e. $T^{\mu\nu}t_\mu$ non-spacelike. $(\rho \geq |p|)$.

(4.5) • NDEC: null dominant energy condition
$T_{\mu\nu}\ell^\mu\ell^\nu \geq 0$ $\forall \ell^\mu$ null vector AND $T^{\mu\nu}\ell_\mu$ non-spacelike.
Same as DEC but allows $p=-\rho$.

(4.6) • SEC : strong energy condition
$T_{\mu\nu}t^\mu t^\nu \geq [(D-2)/2]\, T^\lambda{}_\lambda t^\sigma t_\sigma$ $\forall t^\mu$ timelike
(gravity attractive!) $(p+\rho \geq 0$ and $3p+\rho \geq 0)$

Cosmological Constant / Dark Energy

An especially interesting type of "matter" (so-called) comes from $\Lambda \neq 0$. Recall that

(5.1) $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\Lambda g_{\mu\nu} + T_{\mu\nu} (8\pi G_N)$$

(5.2) If we regard $-\Lambda g_{\mu\nu} = T^{(\Lambda)}_{\mu\nu} \cdot 8\pi G_N$

then it has some unfamiliar properties, e.g.

(5.3) $$8\pi G_N T^{\lambda}{}_{\lambda} = -\Lambda D < 0 \quad (\text{if } \Lambda > 0!)$$

(5.4) and $$T^{(\Lambda)}_{\mu\nu} = -\frac{\Lambda}{8\pi G_N} g_{\mu\nu}$$

(5.5) Comparing to a perfect fluid $T^{(p.f.)}_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}$ in rest frame where $(u^\mu) = (1, 0, 0, 0)\sqrt{-g_{00}}$,

(5.6) we have $(T^{\mu}{}_{\nu}{}^{(p.f.)}) = \mathrm{diag}(-\rho, p, p, p)$

so
$$T^{\mu}{}_{\nu}{}^{(\Lambda)} = -\frac{\Lambda}{8\pi G_N} (\delta^\mu{}_\nu) \quad \text{so}$$

(5.7) $$p = \underset{= \text{constant}}{-\rho} \quad \text{for cosmological constant}$$

Problems: • Λ estimated from particle physics* is very far off from experiment ☹

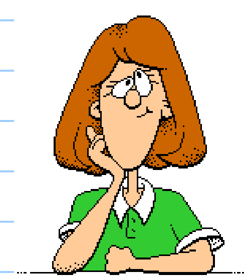
EXCITEMENT

• Not known what is composition of dark energy‡

* $[\Lambda] = m^4$; so $\Lambda \sim k^4_{uv} \sim \frac{1}{l^4_{uv}} \sim m^4_{Pl}$

$\sim (10^{19}\ \text{GeV})^4$

‡ Expt : $\Lambda \sim (10^{-12}\ \text{GeV})^4$



Alternative theories of gravity

Einstein gravity is well-tested, but not religion!

We could have in addition

- a non-Christoffel connection ;
- higher-order R invariants in S_{grav} ;
- gravitational scalar fields ;
- extra dimensions of space ; (etc!)

Scalar-tensor theory*

(6.1) $$S_{grav}[\lambda, g_{\mu\nu}] = \int d^D x \sqrt{-g} \left\{ f(\lambda)\, R_g - \tfrac{1}{2} h(\lambda)\, g^{\mu\nu} \partial_\mu \lambda\, \partial_\nu \lambda - U(\lambda) \right\}$$

How does changing the gravitational action affect the equations of motion for $g_{\mu\nu}$?

See e.g. Carroll pp 182-3 to derive

(6.2) $$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \frac{1}{f(\lambda)} \left[\tfrac{1}{2} T_{\mu\nu}^{(\text{matter})} + \tfrac{1}{2} T_{\mu\nu}^{(\lambda)} + \nabla_\mu \nabla_\nu f - g_{\mu\nu} \Box f \right]$$

where

(6.3) $$T_{\mu\nu}^{(\lambda)} = h(\lambda)\, \nabla_\mu \lambda\, \nabla_\nu \lambda - g_{\mu\nu} \left[\tfrac{1}{2} h(\lambda) (\nabla \lambda)^2 + U(\lambda) \right]$$

and

(6.4) $$h \Box \lambda + \tfrac{1}{2} h' g^{\mu\nu} \nabla_\mu \lambda \nabla_\nu \lambda - U' + f' R = 0 \qquad \left(' \equiv \frac{d}{d\lambda} \right)$$

The story can be rewritten with

(6.5) $$\tilde{g}_{\mu\nu} = 16\pi \tilde{G}\, f(\lambda)\, g_{\mu\nu} \qquad (\text{“Jordan frame”})$$

whence

(6.6) $$S = \int d^D x \frac{\sqrt{-\tilde{g}}}{16\pi \tilde{G}} \left[R_{\tilde{g}} - \tfrac{3}{2} \tilde{g}^{\rho\sigma} \left(\frac{f'}{f} \right)^2 \tilde{\nabla}_\rho \lambda\, \tilde{\nabla}_\sigma \lambda \right]$$

* Brans-Dicke for $f = \frac{\lambda}{16\pi}$, $h = \frac{\omega_{\kappa}}{8\pi\lambda}$ const., $U(\lambda) = 0$

Extra dimensions

Suppose gravity actually spread out in 5 dimensions, rather than 4. Then

(7.1) $$S_{grav} = \frac{1}{16\pi \tilde{G}_N} \int d^5x \sqrt{-\tilde{g}}\, \tilde{R}_g$$

can be rewritten in terms of $(\{\alpha\} = \{\alpha, I\})$

(7.2) . Taking the metric

$$ds^2 = e^{2\alpha\hat{\chi}} d\hat{s}^2 + e^{2\beta\hat{\chi}} \left(dz + \hat{A}_{\hat{\mu}} dx^{\hat{\mu}}\right)^2 , \tag{2.42}$$

with $\beta = (2-D)\alpha$ and $\alpha^2 = 1/[2(D-1)(D-2)]$ [35] gives

$$\sqrt{-g}R_g = \sqrt{-\hat{g}}\left(R_{\hat{g}} - \tfrac{1}{2}(\partial\hat{\chi})^2 - \tfrac{1}{4}e^{-2(D-1)\alpha\hat{\chi}}F^2\right) , \tag{2.43}$$

where F is the field strength of $\hat{A}$.

Higher-derivative gravity

(7.3) $$S_g = \int d^Dx \sqrt{-g}\left(c_1 + c_2 R + c_3 R^2 + c_4 R^{\mu\nu}R_{\mu\nu} + c_5 g^{\mu\nu}\nabla_\mu R \nabla_\nu R + \cdots\right)$$

We know $c_2 \sim \frac{1}{l_p^{d-2}}$; dimensionally, higher-derivative terms must be down by powers of l_p.

(7.4) So extra terms above affect physics only when $\frac{R}{l_p^2} \gtrsim 1$, etc.

Classical theory <u>no good</u> there, however; ☹ also, higher-derivative theories at any finite order in derivatives have propagating ghosts (negative-norm states!)

String theory

Etc. ☺