Last time

We derived Einstein's equations from an action principle

(1.1) $$\boxed{S_{grav} = \frac{1}{16\pi G_N}\int d^D x\,\sqrt{g}\,(R - 2\Lambda)}$$

and we obtained

(1.2) $$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}}$$

(1.3) with $$\boxed{T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta \mathcal{L}_{matter}}{\delta g^{\mu\nu}}}$$

which enables us to find $T_{\mu\nu}$ for any kind of matter.

Let us now investigate different types and their effect on the development of the universe.

We had
$$T_{\mu\nu}^{(dust)} = \rho u_\mu u_\nu \qquad \text{with } \rho = \delta(x - x(\tau))$$

Let's consider scalars and EM fields.

Scalar field

(1.4) $$\boxed{S_s = \int d^D x\,\sqrt{g}\left(-\frac{1}{2} g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right)}$$

In a Minkowski metric, $g^{\mu\nu} = \eta^{\mu\nu}$,

(1.5) $$S_s\,(\text{mink}) = \int d^D x\left\{\frac{1}{2}\dot\phi^2 - \frac{1}{2}|\vec\nabla\phi|^2 - V(\phi)\right\}$$

(and then $\mathcal{H} = \frac{1}{2}\dot\phi^2 + \frac{1}{2}|\vec\nabla\phi|^2 + V(\phi) \geq 0$ ☺)

We've got

(1.6) $$\mathcal{L}_s = -\frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\sqrt{-g}$$

(Note: On a scalar, $\nabla_\mu\phi = \partial_\mu\phi$ ☺)

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \left(-\frac{1}{2}\sqrt{-g}\, g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi - V(\phi)\sqrt{-g} \right)$$

$$= \left(\frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} \right) \left[(\nabla\phi)^2 + 2V(\phi) \right] + \nabla_\mu \phi \nabla_\nu \phi$$

$$= -\frac{1}{2} g_{\mu\nu} (\nabla\phi)^2 - g_{\mu\nu} V(\phi) + \nabla_\mu \phi \nabla_\nu \phi$$

(2.1) $$\boxed{T_{\mu\nu}^{(scalar)} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla\phi)^2 - g_{\mu\nu} V(\phi)}$$

Electromagnetic Fields

(2.2) $$S_{EM} = \int d^D x \sqrt{-g} \left\{ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right\} \rightarrow \mathcal{H} = \frac{1}{2} (\vec{E}^2 + \vec{B}^2)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

▷ Check on general covariance!

We would prefer "$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$" ...

But $$\nabla_\mu A_\nu - \nabla_\nu A_\mu = (\partial_\mu A_\nu - \Gamma^\lambda{}_{\mu\nu} A_\lambda) - (\partial_\nu A_\mu - \Gamma^\lambda{}_{\nu\mu} A_\lambda)$$

$$= (\partial_\mu A_\nu - \partial_\nu A_\mu) - (\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}) A_\lambda$$

and $\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu}$ so the 2nd term $\equiv 0$

(2.3) so $$\boxed{F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu}$$

Then let's find $T_{\mu\nu}^{(EM)}$ ... !

$$T_{\mu\nu} = \left(\frac{\delta}{\delta g^{\mu\nu}} \left(-\frac{1}{4} \sqrt{-g}\, F^{\alpha\beta} F_{\alpha\beta} \right) \right) \cdot \frac{-2}{\sqrt{-g}}$$

$$= \frac{1}{2\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \left[\sqrt{-g}\, g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} \right]$$

$$= -\frac{1}{4} g_{\mu\nu} F^2 + \frac{1}{2} \left(\delta^\alpha_\mu \delta^\beta_\nu g^{\gamma\delta} + g^{\alpha\beta} \delta^\gamma_\mu \delta^\delta_\nu \right) F_{\alpha\gamma} F_{\beta\delta}$$

$$= -\frac{1}{4} g_{\mu\nu} F^2 + \frac{1}{2} \left(F_\mu{}^\alpha F_{\nu\alpha} + F^\alpha{}_\mu F_{\alpha\nu} \right)$$

(3.1) i.e. $\boxed{T_{\mu\nu} = F_\mu{}^\lambda F_{\nu\lambda} - \frac{1}{4} g_{\mu\nu} F^2}$

Notice that

$$T^\mu{}_\mu = F^{\mu\lambda} F_{\mu\lambda} - \frac{1}{4}(D) F^2$$

$$= \frac{1}{4}(4-D) F^2$$

(3.2) $\Rightarrow$ $\boxed{T^\mu{}_\mu = 0 \text{ in 4d E\&M.}}$

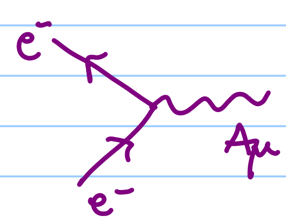
<u>Side remark</u> ← very important physically

Superposition works in E&M because Maxwell's equations $d^2 A = 0$, $*d*dA = J$, are <u>linear</u> in the dynamical variable A.

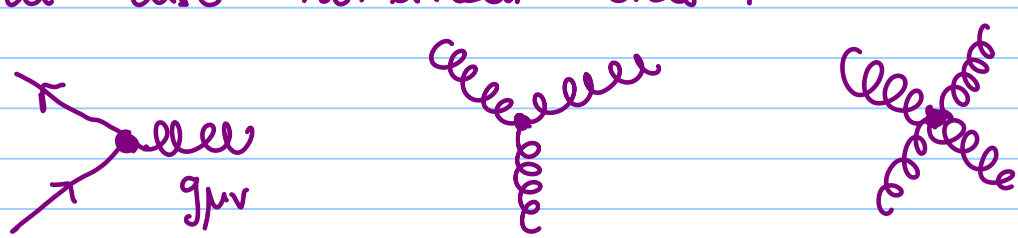
This is to be contrasted with gravity: Einstein's equation involves $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ where R and $R_{\mu\nu}$ are traces of Riemann, which is a nonlinear function of first derivatives of $g_{\mu\nu}$ (via $\Gamma^{\cdot}_{\cdot\cdot}\Gamma^{\cdot}_{\cdot\cdot}$ term) and also of second derivatives of $g_{\mu\nu}$ (via $\partial\Gamma$ term)

(3.3) $\Rightarrow$ $\boxed{\text{Superposition "principle" \underline{FAILS} for GR}}$

In terms of Feynman diagrams, this means that we have not only vertices like the EM one ($\bar{w}\, e^-$)



but also nonlinear ones:



Cosmological Constant / Dark Energy

An especially interesting type of "matter" (so-called) comes from $\Lambda \neq 0$. Recall that

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\Lambda g_{\mu\nu} + T_{\mu\nu}(8\pi G_N)$$

(4.1) If we regard $-\Lambda g_{\mu\nu} = T^{(\Lambda)}_{\mu\nu} \cdot 8\pi G_N$

then it has some unfamiliar properties, e.g.

(4.2) $8\pi G_N T^{\lambda}{}_{\lambda} = -\Lambda D < 0$

(4.3) and $T^{(\Lambda)}_{\mu\nu} = -\frac{\Lambda}{8\pi G_N} g_{\mu\nu}$

Comparing to a perfect fluid $T^{(p.f.)}_{\mu\nu} = (\rho+p)u_\mu u_\nu + p g_{\mu\nu}$ in rest frame where $(u^\mu) = (1,0,0,0)\sqrt{-g_{00}}$, we have $(T^{\mu}{}_{\nu}{}^{(p.f.)}) = \mathrm{diag}(-\rho, p, p, p)$ so

$$T^{\mu}{}_{\nu}{}^{(\Lambda)} = -\frac{\Lambda}{8\pi G_N}(\delta^{\mu}{}_{\nu}) \quad \text{so}$$

(4.4) "$p = -\rho$" for cosmological constant

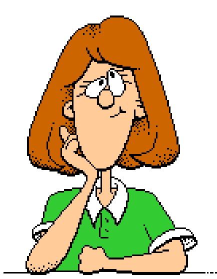
Problems: • Λ estimated from particle physics⊕ is very far off from experiment ☹

$\Uparrow$ EXCITEMENT

• Not known what is composition of dark energy‡

⊕ $[\Lambda] = m^4$; so $\Lambda \sim k_{uv}^4 \sim \frac{1}{l_{uv}^4} \sim m_{Pl}^4$
$\sim (10^{19}\,\mathrm{GeV})^4$

‡ Expt : $\Lambda \sim (10^{-12}\,\mathrm{GeV})^4$ ← How to reconcile ?!?



[[There's ≥ 1 Nobel in this!]]