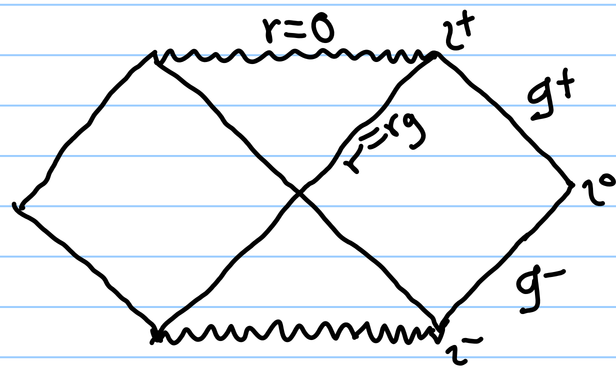
Hawking Radiation: a derivation

Previously, we met simplest BH ie. Schwarzschild.

- First, we studied coörds in which

(1.1) $$ds^2 = -\left(1-\frac{r_g}{r}\right)dt^2 + \left(1-\frac{r_g}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2$$

- Then, looking at causal structure via light-cones, found Kruskal coörds and a maximally analytically extended geometry for which our Penrose diagram was



We had

(1.2) $$ds^2 = \frac{2r_g^3}{r}e^{-(r/r_g)}\left(-d\tilde{t}^2 + d\tilde{r}^2\right) + r^2(\tilde{t},\tilde{r})\, d\Omega_2^2$$

(1.3) where $$\begin{cases} \tilde{t} = \sqrt{r/r_g - 1}\; e^{(r/2r_g)} \sinh(t/2r_g) \\ \tilde{r} = \sqrt{r/r_g - 1}\; e^{(r/2r_g)} \cosh(t/2r_g) \end{cases}$$

Proper distance η defined by ($\Omega_2 = \text{const.}$)

(1.4) $$d\eta^2 = +\left(1-\frac{r_g}{r}\right)^{-1}dr^2$$

(1.5) $$\Rightarrow \eta = \sqrt{r_g(r-r_g)} + r_g \operatorname{arccosh}\left(\sqrt{\frac{r}{r_g}}\right) \underset{r\sim r_g}{\sim} \sqrt{r_g(r-r_g)}$$

Then near horizon

(1.6) $$ds^2 \underset{r\sim r_g}{\sim} \underbrace{-\eta^2 d\omega^2 + d\eta^2}_{\text{Rindler space}} + \underbrace{r_g^2\, d\Omega_2^2}_{\text{transverse 2-sphere}}$$

where $\frac{t}{2r_g} = \omega$

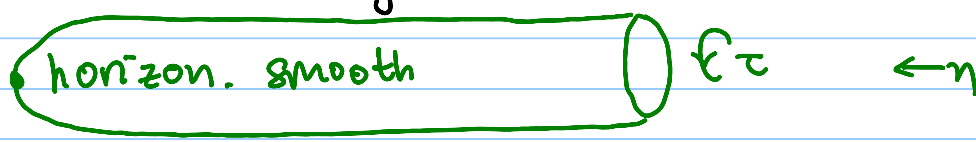
important space-time

Euclidean continuation

Let us Wick rotate $\tau = i\omega$. Then

(2.1) $$ds_E^2 \sim +\eta^2 d\tau^2 + d\eta^2 + r_g^2 d\Omega_2^2$$

Notice : in Euclidean space,
BH has horizon but no singularity
(and no region behind horizon at all!)



* Need to avoid conical singularity in metric
$\Rightarrow$ identify τ to be 2π periodic.

* Path-integral (Lorentzian signature) $\longleftrightarrow$ Thermal partition function (Euclidean)

Period in $\tau_E \leftrightarrow \frac{1}{T_H}$

(2.2) $$\Rightarrow \quad \frac{\hbar}{T_H} = 2\pi \cdot 2r_g = 4\pi r_g \Rightarrow$$

(2.3) $$\boxed{T_H^{(S)} = \frac{\hbar}{4\pi r_g}} \quad \text{Schwarzschild}$$

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Now let's try it for RN

$$ds^2 = -\Delta\, dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega_2^2$$

$$\Delta = 1 - \frac{2GM}{r} + \frac{Q^2}{r^2}$$

Near outer horizon, know

$$\Delta = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right) \approx \left(1 - \frac{r_+}{r}\right)\frac{r_-}{r_+}$$

$$\therefore d\eta^2 = \Delta^{-1} dr^2 = \frac{dr^2\, r^2}{(r-r_+)(r-r_-)} \approx \frac{r^2\, dr^2}{(r-r_+)(r_+-r_-)} = \frac{1}{(r_+-r_-)} \underbrace{\frac{r^2 dr^2}{(r-r_+)}}_{\text{like Schw.}}$$

$$\Rightarrow \quad \eta = \sqrt{(r-r_+)}\, r_+ \cdot \sqrt{\frac{1}{(r_+-r_-)}}$$

So $$ds^2 \simeq +\eta^2 d\tau^2 + d\tilde{\tau}^2 + r_g^2 d\Omega_2$$

so

$$\frac{1}{T_H} = \frac{2\pi\, 2r_+}{\sqrt{r_+ - r_-}} \quad \text{i.e.} \quad T_H^{(RN)} = \frac{\hbar}{4\pi r_+}\sqrt{r_+ - r_-}$$

So as $GM \to |Q|$, $T_H \to 0$

Extremal metric – is zero-temperature
– supports SUSY
– has near-horizon $AdS_2 \times S^2$ throat

Black Hole Mechanics

0) $T_H =$ constant over horizon
1) $dM = TdS -$ (work terms)
2) Area ≥ 0 in classical process

For 1st law $dM = TdS$
Was known firstly that $S \propto$ Area; later Hawking finding $T_H = \frac{\kappa}{2\pi}$ fixed

$$\boxed{S_{BH} = \frac{A}{4G_N}}$$

Can get e.g. by $\int$ up 1st law.

Power radiated?

$$G \cdot -\frac{dM}{dt} \sim (\text{area})(\sigma T^4)$$

Schwarzschild: $$\sim (GM)^2 \left(\frac{1}{(GM)^4}\right) \sim (GM)^{-2}$$

Integrating this i.e.

$$-\frac{G\,dM}{dt} \sim +\frac{1}{(GM)^2},$$

gives

$$\frac{\Delta t}{E_p} \sim (l_p M)^3$$

Long lifetime! If $m_{BH} \sim 10^{15}$ kg then it would be dying today after primordial formation

Killing Horizons

- If any Killing vector becomes null on the horizon, and is null over some hypersurface Σ, then Σ is a Killing horizon

(N.B.: Killing horizons are not necessarily event horizons)

- Classification:
  - Every event horizon Σ in a stationary, asymptotically flat spacetime is a Killing horizon for some vector field $\{\chi^\mu\}$
  - For static spacetimes, $\chi^\mu = (\partial_t)^\mu$ (time transl @ ∞)
  - For stationary spacetimes, have only axisymmetry and Rotational Killing vector field $R^\mu = (\partial_\phi)^\mu$, and then $\chi^\mu = (\partial_t)^\mu + \underbrace{\Omega_H}_{\text{angular speed @ horizon}} (\partial_\phi)^\mu$

- Having a Killing horizon requires some nontrivial symmetry (i.e. KV(s).)

* Sometimes Killing horizons may not be too interesting. Consider e.g. Minkowski

$$ds^2 = -dt^2 + dx^2 + dy + dz^2$$

Boost K.V. is $\chi = x\partial_t + t\partial_x$; norm is $\chi_\mu \chi^\mu = -x^2 + t^2$
goes null @ $x = \pm(c)t$.

More K.H. exist in Minkowski – try repeating the above for other boosts, the rotations, the translations,...

Surface gravity

Consider a Killing horizon and normal vector to Σ.
Along KH, χ^μ obeys geodesic eqn:

(5.1) $$\boxed{(\chi^\mu \nabla_\mu)\chi^\nu = -\kappa \chi^\nu}$$

(integral curves of χ^ν not necessarily affinely parametrized)

- Now make use of Killing equation

(5.2) $$\nabla_{(\mu}\chi_{\nu)} = 0$$

and fact that $\chi \perp \Sigma$ so $\chi_{[\mu}\nabla_\nu \chi_{\sigma]} = 0$

(5.3) $$\boxed{\kappa^2 = -\frac{1}{2}\left[(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)\right]\Big|_{\text{horizon}}}$$

(Only) in a static spacetime, κ is acceleration of static observer near-horizon, , as measured by static observer @ ∞.

- For Schwarzschild, suppose $\{\chi^\mu\} = \left\{\frac{\partial}{\partial t}\left(1-\frac{r_g}{r}\right)^{-\frac{1}{2}}, 0, 0, 0\right\}$

$$\kappa^2 = -\frac{1}{2}(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)\Big|_{\text{horizon}}$$
$$= -\frac{1}{2}(\partial_\mu \chi_\nu - \Gamma^\lambda_{\mu\nu}\chi_\lambda)(\partial_\alpha \chi_\beta - \Gamma^\lambda{}_{\alpha\beta}\chi_\lambda) g^{\alpha\mu} g^{\beta\nu}$$

Had, a long time back, nontrivial Christoffels; (see C p.206) we're interested in $\{\Gamma^\mu_{t\nu}\}$

Have $\Gamma^r{}_{tt} = \frac{r_g}{2r^3}(r-r_g)$; $\Gamma^t{}_{tr} = \frac{(r_g/2)}{r(r-r_g)}$

and $\{K^\mu\} = \{1, 0, 0, 0\}$

So $$-2\kappa^2 = (\nabla_\alpha K^\mu)(\nabla^\alpha K_\mu)$$
$$= (\Gamma^\mu_{\alpha\beta} K^\beta)(\Gamma^\nu{}_{\gamma\delta} K^\delta) g^{\alpha\gamma} g^{\beta\delta}$$
$$= -\frac{r_g^2}{4r^4}\Big|_{\text{horizon}} = -\frac{1}{4r_g^2}$$

i.e. $\boxed{\kappa^{(s)} = \frac{1}{2r_g}}$ $\Rightarrow$ $\boxed{T_H = \frac{1}{4\pi r_g}}$

$TdS = dM$ then $\Rightarrow$ $\boxed{S_{BH} = \frac{4\pi r_g^2}{4G_N}}$ $r_g = GM$ (4d)