GR 24 Sep 2004

Note Title 22/09/2004

Last time, we studied (1)

- momentum 4-vector
- constant acceleration
- the Equivalence Principle

and introduced

- spacetime as a manifold
- metric tensor.

Today :-

- Vectors (again)
- Commutator of vectors
- Tensors (again)
- Metric : canonical form, locally inertial cöords
- Causality
- ϵ tensor density

Vectors (again)

Now we can have arbitrary functions on our manifold. For some function $f(x^\mu)$, we have for $x^\mu(\lambda)$

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\lambda} \quad \leftarrow \text{directional derivative of } f \text{ (along } \lambda\text{-direction)}$$

(1.1) or $$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu$$

i.e. $\{\hat{e}_{(\mu)} = \partial_\mu\}$ is a set of basis vectors.

- The tangent space $T_p(M)$ at some point p can actually be thought of as the space of derivatives in this way(!). It is a vector space ... and the Leibniz rule is obeyed.

- Commutator of 2 vector fields : (direction varies over M)

$$[X,Y](f) = X(Y(f)) - Y(X(f))$$

→ linear
→ obeys Leibniz rule

In components,

(1.2) $$\boxed{[X,Y]^\mu = X^\lambda\partial_\lambda Y^\mu - Y^\lambda\partial_\lambda X^\mu}$$

Example of vector field: wind at surface of Earth.
(Math: always have (at least) 2 zeroes on S^2 ... ☺)

$\boxed{\text{Tensors (again)}}$

As before, but this time $\frac{\partial x^{\mu'}}{\partial x^{\nu}} \neq$ constant matrix

Metric tensor non-flat space

(2.1) $\boxed{g_{\mu\nu}}$ our length-measurer, now depends on x^λ i.e. where you are in space-time.

(2.2) "Line element" $\boxed{ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu}$ (ds^2 invariant under coord changes)

- $g_{\mu\nu}$ is symmetric
- inverse metric is denoted $g^{\mu\nu}$ and

(2.3) $$\boxed{g^{\mu\nu} g_{\nu\lambda} = g^\mu{}_\lambda = \delta^\mu{}_\lambda}$$ ← Kronecker delta

- $\boxed{g \text{ is used to raise and lower indices on tensors}}$

e.g. Euclidean $\mathbb{R}^3$: $ds^2 = dx^2 + dy^2 + dz^2$ Cartesian

$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$ } spherical polars

Let's do an explicit $\boxed{\text{example}}$ with $g_{\mu\nu}$.
Suppose we started with Minkowski spacetime with
$$ds^2 = -c^2 dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$
and transformed to polar coords such that $t' = t$
$$x^1 = r\cos\theta\cos\varphi$$
$$x^2 = r\cos\theta\sin\varphi$$
$$x^3 = r\sin\theta$$

Then the invariant line element is

$$ds^2 = -c^2dt^2 + (dr\cos\theta\cos\varphi - r\sin\theta\, d\theta\cos\varphi - r\cos\theta\sin\varphi\, d\varphi)^2$$

$$+ (dr\cos\theta\sin\varphi - r\sin\theta\, d\theta\sin\varphi + r\cos\theta\cos\varphi\, d\varphi)^2$$

$$+ (dr\sin\theta + r\cos\theta\, d\theta)^2$$

$$= -c^2dt^2 + dr^2\left(\cos^2\theta\cos^2\varphi + \cos^2\theta\sin^2\varphi + \sin^2\theta\right)$$

$$+ d\theta^2\left(r^2\sin^2\theta\cos^2\varphi + r^2\sin^2\theta\sin^2\varphi + r^2\cos^2\theta\right)$$

$$+ d\varphi^2\left(r^2\cos^2\theta\sin^2\varphi + r^2\cos^2\theta\cos^2\varphi\right)$$

$$+ dr\,d\theta\left(-r\cos\theta\sin\theta\cos^2\varphi - r\cos\theta\sin\theta\sin^2\varphi + r\sin\theta\cos\theta\right)$$
$$+ d\theta\,dr\,(\;''\;)$$
$$+ dr\,d\varphi\left(-r\cos^2\theta\cos\varphi\sin\varphi + r\cos^2\theta\sin\varphi\cos\varphi\right)$$
$$+ d\varphi\,dr\,(\;''\;)$$
$$+ d\theta\,d\varphi\left(r^2\sin\theta\cos\theta\cos\varphi\sin\varphi - r^2\sin\theta\cos\theta\sin\varphi\cos\varphi\right)$$
$$+ d\varphi\,d\theta\,(\;''\;)$$
$$= -c^2dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

You will get exactly the same answer (try it!) if you use our expression for 2-index tensors

(3.1) $$g_{\mu'\nu'} = \Lambda^\sigma{}_{\mu'}\Lambda^\lambda{}_{\nu'}\, g_{\sigma\lambda}$$

where

(3.2) $$\Lambda^\sigma{}_{\mu'} = \frac{\partial x^\sigma}{\partial x^{\mu'}}$$ and it's usually quicker :)

Note: For some vector, the quantity $\partial_\mu V_\nu$ is <u>not</u> in general a tensor. Why not?

Consider how this transforms under $x \to x'$:

(3.3) $$\partial_\mu V_\nu \to \frac{\partial}{\partial x^{\mu'}} V_{\nu'} = \left(\frac{\partial x^\lambda}{\partial x^{\mu'}}\frac{\partial}{\partial x^\lambda}\right)\left(\Lambda^\sigma{}_{\nu'} V_\sigma\right)$$

$$= \Lambda^\lambda{}_{\mu'}\left(\Lambda^\sigma{}_{\nu'}(\partial_\lambda V_\sigma) + V_\sigma(\partial_\lambda \Lambda^\sigma{}_{\nu'})\right)$$

(3.4) $$\partial_{\mu'} V_{\nu'} = \underbrace{\Lambda^\lambda{}_{\mu'}\Lambda^\sigma{}_{\nu'}\,\partial_\lambda V_\sigma}_{\text{like a tensor}} + \underbrace{\left[\Lambda^\lambda{}_{\mu'}(\partial_\lambda \Lambda^\sigma{}_{\nu'})\right] V_\sigma}_{\text{oops!}}$$ watch for this later :)

Canonical Form

The metric can, by a suitable change of coordinates, be brought to the form

(4.0) $$(g_{\mu\nu}) = \mathrm{diag}(-1, +1, +1, +1) \,.$$

This has "Lorentzian" signature, whereas our old friend δ_{ij} from $\mathbb{R}^3$ has "Euclidean" signature.

Locally Inertial Coords

At any particular point p, you can choose a reference frame in which

$$\partial_{\hat\sigma}\, g_{\hat\mu\hat\nu}\Big|_p = 0$$

BUT this cannot hold beyond first derivatives. We can do this by looking at a Taylor series expansion for the locally inertial coords (call them $\hat{x}^\mu$) in terms of the regular ones; choosing $\hat{x}^\mu\big|_p = x^\mu\big|_p$ for simplicity and expanding in a Taylor series

(4.1) $$x^\mu = \left(\frac{\partial x^\mu}{\partial x^{\hat\nu}}\right)\Bigg|_p x^{\hat\nu} + \text{ higher-order terms}$$

$\uparrow$ (4×4 = 16 independent components)

We also know that $g_{\hat\mu\hat\nu} = \dfrac{\partial x^\lambda}{\partial x^{\hat\mu}} \dfrac{\partial x^\sigma}{\partial x^{\hat\nu}}\, g_{\lambda\sigma}$.

So $g_{\hat\mu\hat\nu}(\hat{x})\big|_p$

(4.2) $$= g_{\hat\mu\hat\nu}\Big|_p + \frac{\partial}{\partial x^{\hat\beta}} g_{\hat\mu\hat\nu}\Big|_p x^{\hat\beta} + \mathcal{O}(\hat{x}^2)$$

$$= \left(\frac{\partial x^\lambda}{\partial x^{\hat\mu}}\frac{\partial x^\sigma}{\partial x^{\hat\nu}}\, g_{\lambda\sigma}\right)\Bigg|_p + \left(\frac{\partial}{\partial x^{\hat\beta}}\left(\frac{\partial x^\lambda}{\partial x^{\hat\mu}}\frac{\partial x^\sigma}{\partial x^{\hat\nu}}\right) g_{\lambda\sigma} + \left(\frac{\partial x^\lambda}{\partial x^{\hat\mu}}\frac{\partial x^\sigma}{\partial x^{\hat\nu}}\left(\frac{\partial}{\partial x^{\hat\beta}}\, g_{\lambda\sigma}\right)\right)\right)\Bigg|_p x^{\hat\beta} + \ldots$$

Let's compare $\mathcal{O}(1)$ and $\mathcal{O}(\hat{x})$ terms here. We get two equations (to this next-to-lowest order)

(5.1) (a) $g_{\hat\mu\hat\nu}\big|_p = \left(\frac{\partial x^\lambda}{\partial x^{\hat\mu}}\frac{\partial x^\sigma}{\partial x^{\hat\nu}} g_{\lambda\sigma}\right)\Big|_p$ and

(5.2) (b) $\left(\partial_{\hat\beta}\, g_{\hat\mu\hat\nu}\right)\big|_p = \left(\partial_{\hat\beta}\left(\frac{\partial x^\lambda}{\partial x^{\hat\mu}}\frac{\partial x^\sigma}{\partial x^{\hat\nu}}\right) g_{\lambda\sigma} + \frac{\partial x^\lambda}{\partial x^{\hat\mu}}\frac{\partial x^\sigma}{\partial x^{\hat\nu}}\partial_{\hat\beta}\, g_{\lambda\sigma}\right)\Big|_p$

▷ For (a), having 16 functions (see (4.1)) is enough to specify $\frac{1}{2!}(4\times 5) = 10$ independent metric components.

The 6 left over are the 6 parameters of the Lorentz group in the locally inertial frame! ☺ (3 rotations & 3 boosts.)

▷ For (b), for 2nd term, we have 4 derivatives of 10 cpts, i.e. 40 independent bits to worry about. Also, for 1st term,

$$\partial_{\hat\beta}\left((\partial_{\hat\mu}x^\lambda)(\partial_{\hat\nu}x^\sigma)\right) = (\partial_{\hat\beta}\partial_{\hat\mu}x^\lambda)(\partial_{\hat\nu}x^\sigma) + (\partial_{\hat\mu}x^\lambda)(\partial_{\hat\beta}\partial_{\hat\nu}x^\sigma)$$

Notice that $\frac{\partial^2 x^\lambda}{\partial x^{\hat\beta}\partial x^{\hat\mu}}$ is symmetric in $(\hat\beta \leftrightarrow \hat\mu)$ so it also has 40 cpts.

So by adjusting ↑ as necessary, we can make $\partial_{\hat\beta}\, g_{\hat\mu\hat\nu} = 0$ at p (ONLY!), as required.

<u>Pattern</u>: at higher orders in Taylor expansion, there are <u>not</u> enough independent components of 3rd derivative of x to adjust 2nd deriv's of g to zero. [This is where a new construct, called the Riemann curvature tensor, which we will meet later, comes in.]

See Carroll p.75 for the proof.

<u>Question</u> :

If special relativity "works" only locally, what happens in curved space-time to our notions of causality?

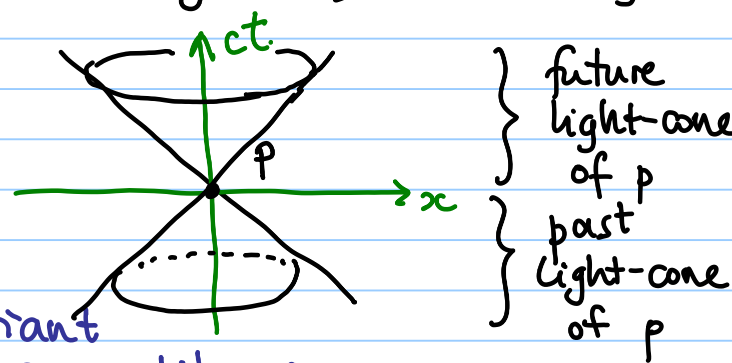
Causality

Recall that in lecture of 15 Sep, I introduced the SR concept of the light-cone. We can use the same idea in GR, but it will depend on our point p.

The future light-cone of p in SR is all the points q that can be reached from p via light-ray. Similarly for the past light-cone.

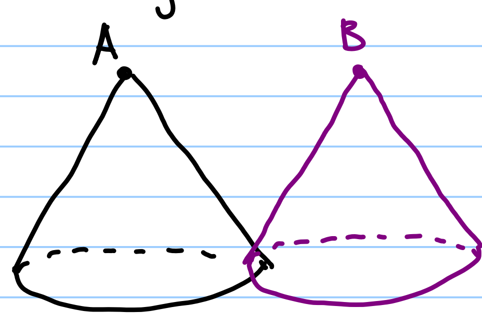
Note: "points" in space-time are also called "events".

SR



Even in SR, there is no invariant notion of simultaneity. But causality is well-defined: if p could have caused q (e.g. by sending a signal using $|v| \leq c$ particle) then q could not have caused p (without violating the law that $|v| \leq c$.)

We can also consider 2 points in Minkowski space, A and B, and ask if they could have been caused by the same event p. We figure this out by looking at their past light-cones:



Going back far enough, the past light-cones overlap.

This is true in Minkowski space for any A & B (!)

Such points are said to be "in causal contact".

Let's do a curved-space example.

(7.1) Consider $ds^2 = -c^2dt^2 + t\,d\vec{x}^2$

This spacetime doesn't make sense for $t \le 0$; in particular, physics is singular at $t=0$ (roughly, ∵ proper lengths in all directions go to zero there.). So restrict $0 < t < +\infty$.

Let a light-ray go along x^3 direction (w.l.o.g.) Then for path $x^\mu(\lambda)$ of light-ray, we have

$$ds^2 = 0 \qquad \text{(light-like interval)}$$
$$= -c^2dt^2 + t(dx^3)^2 + 0 + 0$$

and for this path $\frac{dx^\mu}{d\lambda}$ is a tangent vector so that

$$\left(\frac{ds}{d\lambda}\right)^2 = 0 = -c^2\left(\frac{dt}{d\lambda}\right)^2 + t\left(\frac{dx^3}{d\lambda}\right)^2 \Rightarrow \frac{dx^3}{dt} = \pm\frac{c}{\sqrt{t}} \rightarrow \pm\infty \text{ @ } t\to 0$$

(7.2) c.f. Minkowski-spacetime answer $\frac{dx^3}{dt} = \pm c$.

⇒ light-cones are not generally at 45° on spacetime diagram.

(7.3) We can also integrate: $x^3(t) - x^3_0 = \pm 2c(\sqrt{t} - \sqrt{t_0})$

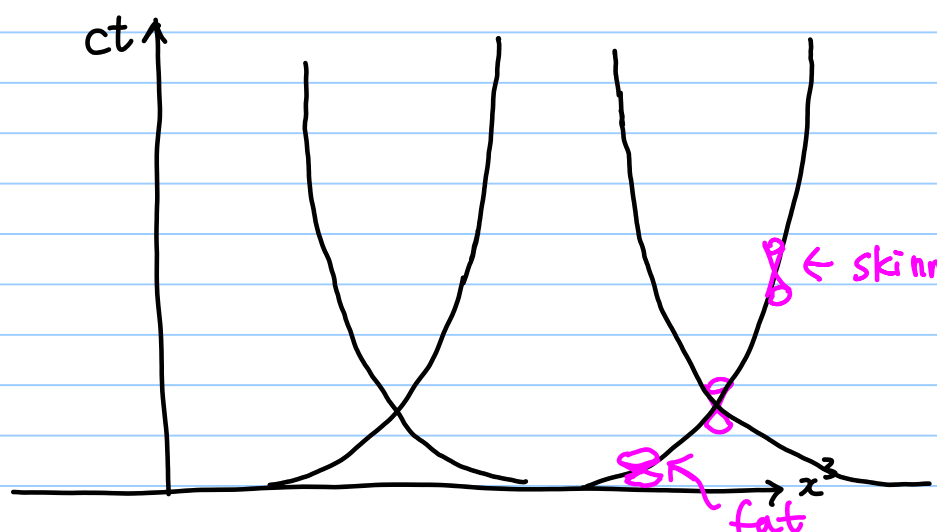
Let's follow 2 particles; assume same clock (OK by (7.1))

$$x^3_{(j)} = x_{0(j)} \pm 2c\sqrt{t} \qquad j=1,2$$

i.e.

(7.4) $$ct_{(j)} = \frac{1}{4}\left(x_{(j)} - x_{0(j)}\right)^2 \qquad t>0$$

← Parabolas



- Light-cones degenerate @ $t \to 0$
- These 2 points are not causally connected (c.f. Big Bang!)