GR 22 Sep 2004

Note Title 21/09/2004

Last time, physics in flat spacetime :- (1)

* Vectors
  how components change under coord changes

* Dual vectors

* Tensors

Today :

(1) continuing physics in flat spacetime

* 4-velocity for massive particles
* combining E and $\vec{p}$ into 4-momentum p^μ

(2) constant acceleration
* the uniformly accelerating astronaut example

(3) Beginnings of "gravity as geometry"

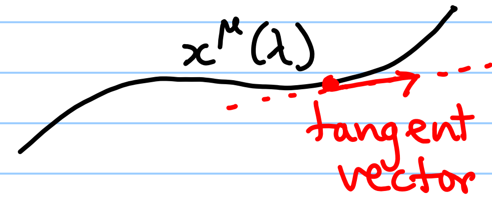
* Equivalence principle

* Spacetime as a manifold

* The metric tensor ($g_{\mu\nu}$)

Momentum 4-vector

Consider a particle moving along a "world-line"



Map from $\mathbb{R}$ to $\mathbb{R}^{1,3}$
λ x^μ

valid for massive particle

If we use proper time τ for λ, then the tangent vector is

(2.1) $$\boxed{\frac{dx^\mu}{d\tau} = u^\mu}$$ ← 4-velocity.

The components u^μ make up a 4-vector because the proper time is defined according to

(2.2) $$-c^2 d\tau^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$$ (∗) (dimensions of length)

and because x^μ is a 4-vector.

if $m^2 > 0$

(2.3) Then the 4-momentum is $\boxed{p^\mu = m u^\mu}$ $\boxed{(p^\mu) = (\frac{E}{c}, \vec{p})}$

Even for massless particles, p^μ still makes sense but is not m times u^μ!

Note that u^μ is unit normalized: (dimensionless)

$$u^\mu u_\mu = \eta_{\mu\nu} u^\mu u^\nu = \eta_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1 \quad \text{by } (∗)$$

Check units

So we obtain

(2.4) $$\boxed{p^\mu p_\mu = -m^2 c^2}$$ Works for $m^2 > 0$ and $m^2 = 0$.

$m^2 > 0$ In the particle's rest frame,

$$(p^\mu)\big|_{rest} = (mc, \vec{0})$$ and in another frame

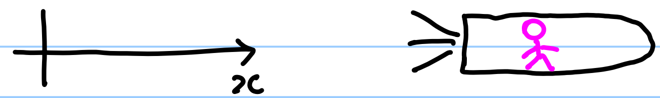
$$(p^\mu) = (\gamma mc, \gamma m \vec{v}), \quad \gamma = 1/\sqrt{1 - v^2/c^2}.$$

$m^2 = 0$ $$(p^\mu) = (p, -p, 0, 0).$$

The constant-acceleration astronaut

When I was an undergrad, a prof. introduced the idea of the twin paradox – could the space-traveller twin really live longer by travelling at relativistic speeds? – but never equipped me with the technology to answer this question (!!). Here is how you can solve that.

- Consider a spaceship moving along the x direction in our stationary ("laboratory") frame.



constant acceleration g

The astronaut has a watch and measures proper time τ.

▷ How do we cope with the fact that this is not an inertial frame, i.e. not moving @ constant velocity?

* Approximate, at each instant of time, by an inertial frame of reference at that speed v.

Mathematically, it is the rapidity ζ which is additive i.e. under 2 successive Lorentz transformations it is ζ that adds rather than velocities.

Infinitesimal addition to rapidity $d\zeta$ given in terms of acceleration g and proper time τ, because of our successive approximations by locally inertial frames

(3.1) Units? $\quad \dfrac{v}{c} = \tanh\zeta \quad$ i.e. ζ is dimensionless.

acceleration: $[a] = \dfrac{L}{T^2}$

proper time $[\tau] = T$

so $[a\tau] = \dfrac{L}{T} = [c]$

(4.1) Then
$$d\zeta = \frac{g}{c}\, d\tau$$

Each infinitesimal tick in astronaut proper time leads to an increase in rapidity measured in our/lab frame. Since g is constant,

(4.2)
$$\boxed{\zeta = \frac{g}{c}\tau}$$

Now we'd like to compute the distance in lab frame moved during lab time dt. This is simply given by the speed

(4.3)
$$dx = v\, dt = c \tanh\zeta\, dt$$

Next, to convert to astronaut time we use (4.2), and the time dilation formula

(4.4)
$$dt = \frac{d\tau}{\sqrt{1 - v^2/c^2}} = \cosh\zeta\, d\tau$$

i.e.
$$dx = c \tanh\zeta \cosh\zeta\, d\tau$$

(4.5)
$$\Rightarrow \quad dx = c \sinh\zeta\, d\tau$$

This we can integrate! Assuming that $x(\tau=0)=0$, we have
$$x(\tau) = c\int d\tau \sinh\left(\frac{g\tau}{c}\right)$$

(4.6)
$$\Rightarrow \quad \boxed{x(\tau) = \frac{c^2}{g}\left[\cosh\left(\frac{g\tau}{c}\right) - 1\right]}$$

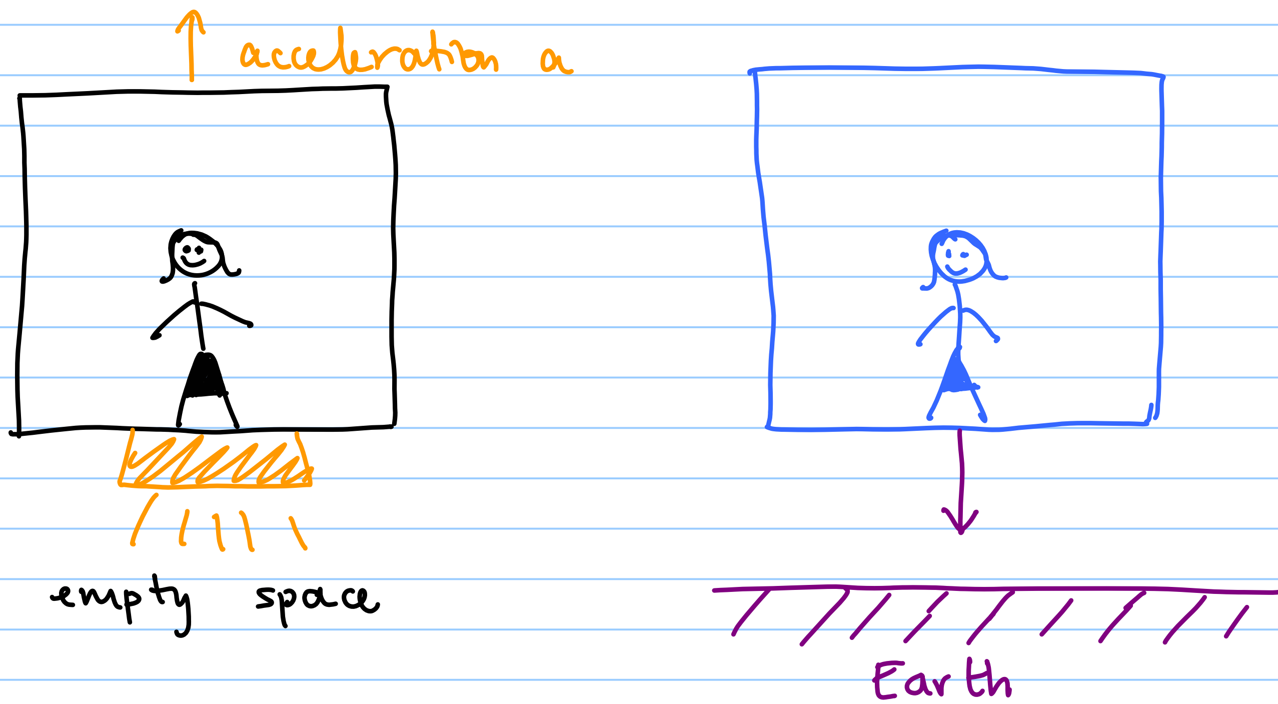
Using this, for a constant acceleration you can figure out the distance the astronaut travels as a function of astronaut (proper) time. In particular, you can compute what happens with a path
- (i) constant speed $+v$ towards a star;
- (ii) constant deceleration, until speed $=-v$;
- (iii) constant speed $-v$ back home.

Equivalence Principle

Einstein thought super-hard about what constant acceleration feels like. He thought a lot about elevators in particular, in this context.

Question: does acceleration feel like gravity?

VOTE.



Comparison: $\vec{E}$ field

where $\vec{E} = \frac{Q}{r^2}\hat{r}$ depends on sign of Q

so that $\vec{F}_{el} = -\frac{Qq}{r^2}\hat{r}$ for test-charge q

$\vec{F} = m\vec{a} \Rightarrow$ physics depends on m/q. can be ⊕ or ⊖ even for antimatter!

Gravity, by contrast, is always attractive.

In fact, what Einstein realized is that gravity is exactly like acceleration; (techically this holds true when a non-inertial frame is approximated locally and instantaneously).

In equations: $\vec{F} = m_i \vec{a}$ (or relativistic generalization) inertial mass

and

$\vec{F} = -m_g \vec{\nabla}\phi$ for some potential ϕ gravitational mass

e.g. for spherical mass $\vec{F} = \left(-\frac{GM}{r^2}\hat{r}\right) m_g$

or in constant field $\vec{F} = m_g \vec{g}$

(6.1) and $\vec{a}$ is like $\vec{g}$ if $\boxed{m_{inertial} = m_{gravitational}}$

This is called the (weak) Equivalence Principle

and has held up under experimental testing for the best part of an entire century.

The basic idea can also be expressed relativistically in a simple way as the Einstein Equivalence Principle or EEP [Carroll] by saying

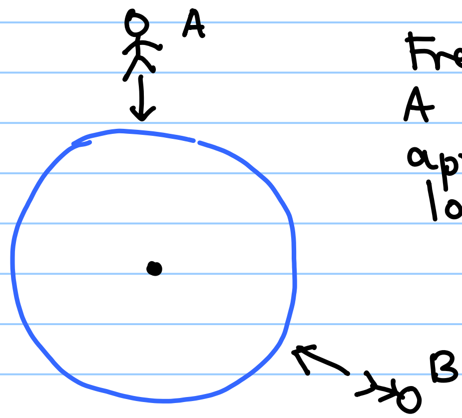
> It is impossible to detect the presence of gravity via local experiments (at a point); i.e. every reference frame can be instantaneously and locally approximated by a Lorentz frame i.e. by special relativity.

This is extraordinarily important to GR – it says: locally in spacetime, everything is just special relativity!

We want to handle the whole banana. ☺ ⇒ let's go!

Consequences

1. Consider gravitational field of Earth.

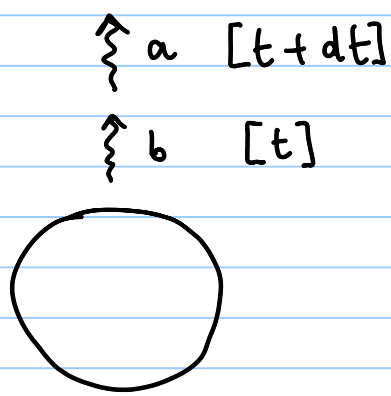


Free-falling observers A and B can be approximated by local Lorentz frames
BUT
these must be different frames.

$\Rightarrow$ non-local experiments can show gravity.

2. Suppose instead of an astronaut we had a photon.
Acceleration $\Rightarrow$ at lab time $t + dt$, it goes slower than at time t
$\Rightarrow$ less Doppler shift.

So in gravitational field of Earth:



photon has lower speed at a than at b
so less Doppler shift;
$\Rightarrow$ photon suffers a redshift in going from a to b

$\Rightarrow$ clocks run slower deeper in a gravitational field.

GPS uses these sorts of facts DAILY.

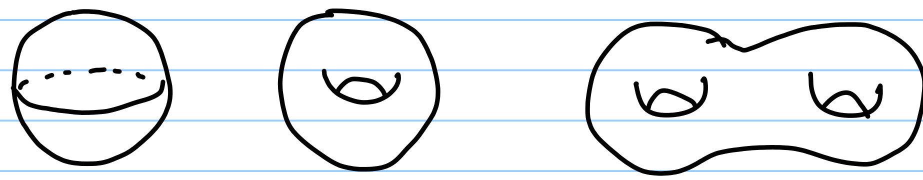
"GR in everyday modern rich-human life!"

Gravity as Geometry

Basic idea: spacetime as a manifold.

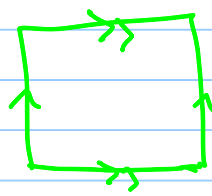
may be curved, or have nontrivial topology, but locally it looks like $\mathbb{R}^n$.

e.g. 2-d:

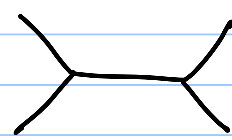


handles = "genus" g

from



Not a manifold:



1-d

2-d

Not a differentiable manifold:

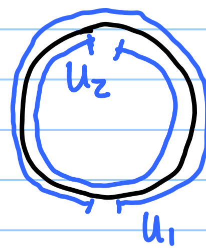
Cone [Dunce cap]

Manifold with boundary (ies):

See Carroll for full mathematical definition of a manifold. Here, the following will suffice:–

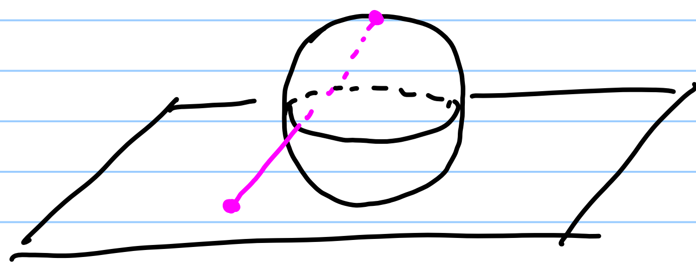
A manifold is a space[-time] that has coordinate patches on it; and in the transition regions between any two patches, things are sewn together nicely with the transition functions.

e.g. on S^1



Can't cover whole thing with 1 coordinate chart $\because$ would not be open set in $\mathbb{R}$

e.g. S^2 need some projection to get onto $\mathbb{R}^2$ like a piece of paper (a map) e.g. Mercator projection misses N & S poles; see p.61 of Carroll for stereographic projection that misses one pole



Vectors (again)

Now we can have arbitrary functions on our manifold. For some function $f(x^\mu)$, we have for $x^\mu(\lambda)$

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\lambda}$$ ← directional derivative of f (along λ-direction)

(9.1) or $$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu$$

i.e. $\{\hat{e}_{(\mu)} = \partial_\mu\}$ is a set of basis vectors.

- The tangent space $T_p(M)$ at some point p can actually be thought of as the space of derivatives in this way(!). It is a vector space ... and the Leibniz rule is obeyed.

Example of vector field:
Wind direction at surface of Earth
Note: has ≥ 2 zeroes! (math ☺)

$\boxed{\text{Tensors (again)}}$

As before, but this time $\frac{\partial x^{\mu'}}{\partial x^{\nu}} \neq$ constant matrix

Metric tensor non-flat space

(10.1) $\boxed{g_{\mu\nu}}$ our length-measurer, now depends on x^λ i.e. where you are in space-time.

(10.2) "Line element" $\boxed{ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu}$ (ds^2 invariant under coord changes)

- $g_{\mu\nu}$ is symmetric
- inverse metric is denoted $g^{\mu\nu}$ and

(10.3) $$\boxed{g^{\mu\nu} g_{\nu\lambda} = g^\mu{}_\lambda = \delta^\mu{}_\lambda}$$ ← Kronecker delta

- $\boxed{g \text{ is used to raise and lower indices on tensors}}$

e.g. Euclidean $\mathbb{R}^3$: $ds^2 = dx^2 + dy^2 + dz^2$ Cartesian
$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$ spherical polars