Last time we started on deriving Einstein's equations for GR from an action principle.

(1.1) • Experiment $\Rightarrow$ massless, spin-two field (rather than some other type of field.) $\boxed{g_{\mu\nu}}$

• Lowest-order-in-derivatives action consistent with symmetries (and Occam's Razor) $\Rightarrow$

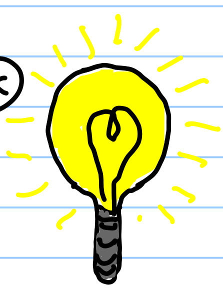
(1.2)
$$\boxed{S_{grav} = \frac{1}{16\pi G_N}\int d^Dx \sqrt{-g}\,(R - 2\Lambda)}$$

We began to find eqns of motion by first focusing on

(1.3)
$$\delta\sqrt{-g} = -\tfrac{1}{2}\sqrt{-g}\, g_{\mu\nu}\,\delta g^{\mu\nu} \quad \left(= +\tfrac{1}{2}\sqrt{-g}\, g^{\mu\nu}\,\delta g_{\mu\nu}\right)$$

The trickiest part is finding δR, and we can simplify that by writing

$$\delta(R) = \delta(g^{\mu\nu} R_{\mu\nu})$$
$$= \underbrace{(\delta g^{\mu\nu})}_{\text{this we know}} R_{\mu\nu} + g^{\mu\nu}\underbrace{(\delta R_{\mu\nu})}_{\text{this we want}}$$



* Easier (MUCH!) to find $\frac{\delta R_{\mu\nu}}{\delta \Gamma^{\alpha}{}_{\beta\gamma}}$ than to find $\frac{\delta R_{\mu\nu}}{\delta g_{\alpha\beta}}$
largely because $\Gamma^{\cdot}_{\cdot\cdot}$ is first-order in derivatives of $g_{\cdot\cdot}$ while $R_{\cdot\cdot}$ is second-order.

The only potential fly in the ointment here is: what if the fantasy that $\delta\Gamma^{\cdot}_{\cdot\cdot}$ is independent of the $\delta g_{\cdot\cdot}$ is actually just that: fantasy ??

The super-cool observation of Palatini was that insisting on a torsion-free, metric-compatible connection does not bugger up anything $\binom{\nabla}{\delta}$. ☺

Some authors of GR texts call this procedure "letting $\delta\Gamma$ flap in the breeze" and I like this terminology. More accurately, it's called using the "first-order formalism"; insisting $\nabla_\alpha g_{\beta\gamma} = 0$ after the fact just neatly connects $\Gamma^\alpha{}_{\beta\gamma}$ to $g_{\mu\nu}$ (in the way we know).

OK. So let's go compute $\dfrac{\delta R^\alpha{}_{\beta\gamma\delta}}{\delta\Gamma^\mu{}_{\nu\lambda}}$!

Possible worry: $\Gamma^\mu{}_{\nu\lambda}$ is NOT a tensor (as we saw in lecture of 29 Sep, there are terms in the transformation law for $\Gamma^\mu{}_{\nu\lambda}$ which spoil tensoriality...).
Resolution: interestingly, these "nasty bits" actually go away if you take a difference of two Γ's. In particular:

(2.1) $\boxed{\delta\Gamma^\mu{}_{\nu\lambda} \text{ is a tensor.}}$ (Try it & see!)

- We're interested in $R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha{}_{\delta\beta} + \Gamma^\alpha{}_{\gamma\epsilon}\Gamma^\epsilon{}_{\delta\beta} - (\gamma\leftrightarrow\delta)$

By taking
$$\nabla_\gamma(\Gamma^\alpha{}_{\delta\beta}) = \partial_\gamma(\Gamma^\alpha{}_{\delta\beta}) + \Gamma^\alpha{}_{\gamma\sigma}(\delta\Gamma^\sigma{}_{\delta\beta}) - \Gamma^\sigma{}_{\gamma\delta}(\delta\Gamma^\alpha{}_{\sigma\beta}) - (\gamma\leftrightarrow\delta)$$

it is straightforward* to find that

(2.2) $$\boxed{\delta R^\alpha{}_{\beta\gamma\delta} = \nabla_\beta(\delta\Gamma^\alpha{}_{\delta\beta}) - \nabla_\delta(\delta\Gamma^\alpha{}_{\beta\gamma})}$$

bona fide tensorial equation

* if rather tedious ... ☺

Now we can finally write δS_{grav} :-

(3.1) $$\delta S_{grav} = \frac{1}{16\pi G_N} \int d^D x \left\{ (\delta\sqrt{-g})\, g^{\alpha\beta} R_{\alpha\beta} + \sqrt{-g}\,(\delta g^{\alpha\beta}) R_{\alpha\beta} + \sqrt{-g}\, g^{\alpha\beta} (\delta R_{\alpha\beta}) - 2\Lambda\, \delta(\sqrt{-g}) \right\}$$

(3.2) $$= \frac{1}{16\pi G_N} \int d^D x \left\{ \left(-\tfrac{1}{2}\sqrt{-g}\, g_{\mu\nu} R + \sqrt{-g}\, R_{\mu\nu}\right) \delta g^{\mu\nu} + \Lambda \sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu} + \sqrt{-g}\, g^{\alpha\beta} g^{\gamma\delta} \left[\nabla_\gamma (\delta \Gamma^\alpha{}_{\delta\beta}) - \nabla_\delta (\delta\Gamma^\alpha{}_{\beta\gamma}) \right] \right\}$$

Now, since

(a) metric is compatible with connection $\Rightarrow \nabla_\alpha g_{\beta\gamma} = 0$

(b) for any vector V^β, $\int d^D x \sqrt{-g}\, \nabla_\mu V^\mu = \int d^D x\, \partial_\mu V^\mu$

(see Wed 29 Sep. lecture notes)

i.e. get total derivative = irrelevant on a topologically trivial spacetime (we will assume this here.)

(3.3) So $$\boxed{\delta S_{grav} = \frac{1}{16\pi G_N} \int d^D x \sqrt{-g} \left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right) \delta g^{\mu\nu}}$$

Now, typically S_{matter} does involve $g^{\alpha\beta}$, even if only to contract up some vector or tensor indices to make a scalar!

So we derive that, with

(3.4) $$\boxed{T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta(\mathcal{L}_{matter})}{\delta g^{\mu\nu}}}$$

(3.5) $$\boxed{R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}}$$

VERY important equations, GR in a nutshell!

cosmological constant (Einstein's "biggest mistake of my life") (not!!)

Example of $T_{\mu\nu}$: particles

We had the diffeomorphism-invariant action

(4.1) $$S_{particle} = -\int d\lambda \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

where $x^\mu(\lambda)$ and $\dot{x}^\mu = \frac{dx^\mu(\lambda)}{d\lambda}$,

$\lambda =$ affine parameter ($\lambda = c_1\tau + c_2$ for $m^2>0$)

Let's do this case.

(4.2) We had $T_{\mu\nu}(x) \equiv \frac{-2}{\sqrt{-g(x)}} \frac{\delta \mathcal{L}_{matter}}{\delta g^{\mu\nu}(x)}$

where

$$S_{matter} = \int d^Dx \, \mathcal{L}_{matter}$$

To have diff invariance, have

$$S_m = \int d^Dx \sqrt{-g} \, \hat{\mathcal{L}}_m$$

SCALAR

Then (7.2) becomes

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta \hat{\mathcal{L}}_m \sqrt{-g}}{\delta g^{\mu\nu}}$$

$$= \frac{-2}{\sqrt{-g}} \left\{ \left(\frac{-1}{2}\sqrt{-g}\, g_{\mu\nu}\right) \cdot \hat{\mathcal{L}}_m + \sqrt{-g} \frac{\delta \hat{\mathcal{L}}_m}{\delta g^{\mu\nu}} \right\}$$

(4.3) $$\Rightarrow \boxed{T_{\mu\nu} = g_{\mu\nu}\hat{\mathcal{L}}_m - \frac{2\,\delta\hat{\mathcal{L}}_m}{\delta g^{\mu\nu}} \quad \text{with} \quad S_m = \int d^Dx \sqrt{-g}\, \hat{\mathcal{L}}_m}$$

← important, and a tensor equation.

Ummm... so how do we handle the problem that $S_{particle}$ is not written as $\int d^Dx \sqrt{-g}$ (something)?

Well, we can do a mathematical trick by

(4.4) inserting into L_{matter}

$$1 = \int d^Dy \; \delta^D(y - x(\lambda))$$

!

OK. So

$$S_M = -\int d\tau \left\{ \int d^D y \; \delta^D(y - x(\tau)) \right\} \sqrt{-g_{\mu\nu}(x(\tau))\,\dot{x}^\mu \dot{x}^\nu}$$

$$= \int d^D y \left[-\int d\tau \sqrt{-g_{\mu\nu}(x)\,\dot{x}^\mu \dot{x}^\nu}\; \delta^D(y - x(\tau)) \right]$$

(5.1) $\Rightarrow$ $$\mathcal{L}_m(y) = -\int d\tau \sqrt{-g_{\mu\nu}(x)\,\dot{x}^\mu \dot{x}^\nu}\; \delta^D(y - x(\tau))$$

So $$T_{\mu\nu}(y) = \frac{-2}{\sqrt{-g(y)}} \frac{\delta \mathcal{L}_m(y)}{\delta g^{\mu\nu}(y)}$$

Notice that

(5.2) $$g_{\mu\nu}(x)\,\dot{x}^\mu \dot{x}^\nu = g^{\mu\nu}\,\dot{x}_\mu \dot{x}_\nu \quad . \text{ So}$$

$$T_{\mu\nu}(y) = -\frac{2}{\sqrt{-g(y)}} \left[-\int d\tau \, \frac{1}{2} \left(-\dot{x}^2\right)^{-1/2} \cdot -\dot{x}_\mu \dot{x}_\nu \cdot \delta^D(y - x(\tau)) \right]$$

(5.3) $$\boxed{T_{\mu\nu}(y) = \int d\tau \; \frac{\delta^D(y - x(\tau))}{\sqrt{-g(y)}} \; \frac{\dot{x}_\mu(\tau)\,\dot{x}_\nu(\tau)}{\sqrt{-\dot{x}^2(\tau)}}}$$

(5.4) Compare $T_{\mu\nu}(\text{dust}) = \rho\, u_\mu u_\nu$

(5.5) identifying $\dot{x}_\mu(\tau) = u_\mu$, we have

$$\rho(y) = \int d\tau \; \frac{\delta^D(y - x(\tau))}{\sqrt{-g(y)}} \; \frac{1}{\sqrt{-u^\mu u_\mu}}$$

(5.6) i.e. $$\boxed{\rho(y) \underset{\text{e.o.m.}}{=} \int d\tau \; \frac{\delta^D(y - x(\tau))}{\sqrt{-g(y)}}}$$ and $$\boxed{T_{\mu\nu} = \underbrace{\rho}_{\text{scalar } (\phi)} \underbrace{u_\mu u_\nu}_{\text{vectors}}}$$

for a particle that obeys its equations of motion (4-velocity squares to -1 … etc.)

so

(ϕ) $$\rho(y) \underset{\text{e.o.m.}}{=} \underbrace{\int d\tau}_{\text{scalar}} \; \frac{\delta^D(y - x(\tau))}{\sqrt{-g(y)}}$$

This guy is a bona fide scalar; δ alone is NOT.

$$1 = \int \left(d^D y \sqrt{-g}\right) \left(\frac{\delta^D(y)}{\sqrt{-g}}\right)$$

So $\rho(y)$ = where the particles are!

Makes full sense.