# REISSNER–NORDSTRØM = BLACK HOLE #2

⊛ Black hole menagerie is bigger than {Schwarzschild} ☺!

One other famous solution with event horizon is BH solving EM + GR. Suppose started from scratch ?

- Knowing that $A_{\mu,\nu} - A_{\nu,\mu} = A_{\mu;\nu} - A_{\nu;\mu}$ because $F = dA$ is antisymmetric

$\boxed{{}_{,\mu} \equiv \partial_\mu}$ $\boxed{{}_{;\mu} \equiv \nabla_\mu}$

$\Rightarrow$ $F_{\mu\nu}F^{\mu\nu}$ is a bona fide scalar so

(1.1) $$\boxed{S_{GR+EM} = \frac{1}{16\pi G_D}\int d^Dx\sqrt{-g}\left(R - \frac{g_0^2}{4\pi}F^{\mu\nu}F_{\mu\nu}\right)}$$

This has e-o-m

(1.2) $$\boxed{R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = (g_0^2/8\pi)\,T^{(F)}_{\mu\nu}}$$

where

(1.3) $$\boxed{T^{(F)}_{\mu\nu} = \left(F_\mu{}^\lambda F_{\nu\lambda} - \tfrac{1}{4}g_{\mu\nu}F^2\right)}$$

and

(1.4) $$\boxed{\nabla_\mu F^{\mu\nu} = 0}$$

How would we solve this ?

(1.5) Consider $\nabla_\mu F^{\mu\nu} = 0 = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,F^{\mu\nu}\right)$

(1.6) and $d\wedge F_{(2)} = 0$.

(1.7) d=4, & $\boxed{\text{Spherical symmetric \& static}}$ ⇒ $\boxed{\text{Killing vectors } \boxed{\partial_t \,\&\, \partial_\varphi}}$

(1.8) Then (1.6) & (1.7) ⇒ $\boxed{F_{(2)} = F_{tr}(r)\,dt\wedge dr + F_{\theta\varphi}(\theta)\,d\theta\wedge d\varphi}$

Immediately satisfies $d\wedge F_{(2)} = 0$

Turn to $\nabla_\mu F^{\mu\nu} = 0$ to solve eqns for $F_{tr}$ & $F_{\theta\varphi}$.
Have many ways to approach this. One is via forms.
Find Hodge dual : $*F$, satisfying $*d*F = 0$ or $d(*F) = 0$

(1.9) And know $(*F)_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}F^{\lambda\sigma}$

(2.1) $$(*F)_{\mu\nu} = \frac{1}{2}\tilde{\epsilon}_{\mu\nu\lambda\sigma}\sqrt{-g}\, g^{\lambda\alpha} g^{\sigma\beta} F_{\alpha\beta}$$

(2.2) Let $ds^2 = -e^{2\alpha}dt^2 + e^{2\beta}dr^2 + r^2 d\Omega_2^2$

(2.3) $$\Rightarrow \sqrt{-g} = e^{(\alpha+\beta)} r^2 \sin\theta$$

$$(*F)_{tr} = \tilde{\epsilon}_{tr\theta\varphi}\left(e^{(\alpha+\beta)}r^2\sin\theta\right)\frac{1}{r^2}\frac{1}{r^2\sin^2\theta}F_{\theta\varphi}$$

(2.4) i.e. $$\boxed{(*F)_{tr} = \frac{e^{(\alpha+\beta)}}{r^2\sin\theta}F_{\theta\varphi}(\theta)}$$

while

$$(*F)_{\theta\varphi} = -\tilde{\epsilon}_{\theta\varphi tr}\left(e^{(\alpha+\beta)}r^2\sin\theta\right)e^{-2\alpha}e^{-2\beta}F_{tr}(r)$$

(2.5) i.e. $$\boxed{(*F)_{\theta\varphi} = -e^{-(\alpha+\beta)}r^2\sin\theta\, F_{tr}}$$

(2.6) Need $d*F = 0$. Have

$$(*F)_{(2)} = -e^{-(\alpha+\beta)}r^2\sin\theta\, F_{tr}\, d\theta\wedge d\varphi + e^{+(\alpha+\beta)}(r^2\sin\theta)^{-1}F_{\theta\varphi}\, dt\wedge dr$$

$$\Rightarrow d(*F)_{(2)} = -\partial_r\left[e^{-(\alpha+\beta)}r^2F_{tr}(r)\right]\sin\theta\, d\theta\wedge d\varphi + \partial_\theta\left[(\sin\theta)^{-1}F_{\theta\varphi}\right]e^{(\alpha+\beta)}r^{-2}\, dt\wedge dr \equiv 0$$

(2.7) $$\Rightarrow \boxed{F_{tr}(r) = \frac{Q}{4\pi g_0}\frac{e^{(\alpha+\beta)}}{r^2}}$$

(2.8) and $$\boxed{F_{\theta\varphi}(\theta) = \frac{P(\sin\theta)}{4\pi g_0}}$$ ← {for the brave at heart (it's a magnetic monopole !!)}

Solving the Einstein equations gives

$\alpha+\beta=0$

and thence

(2.9) $$\boxed{ds^2_{RN} = -\Delta\, dt^2 + \Delta^{-1}dr^2 + r^2 d\Omega_2^2}$$

(2.10) $$\boxed{\Delta = 1 - \frac{2GM}{r} + \frac{(Q^2+P^2)}{r^2}}$$

[Caroll sets $\frac{g_0^2}{4\pi} = 16\pi G$ ...]

This has horizons at

(3.1) $$\boxed{r_{\pm} = GM \pm \sqrt{G^2M^2 - (Q^2+P^2)}}$$

Extremal situation when $r_+ = r_-$

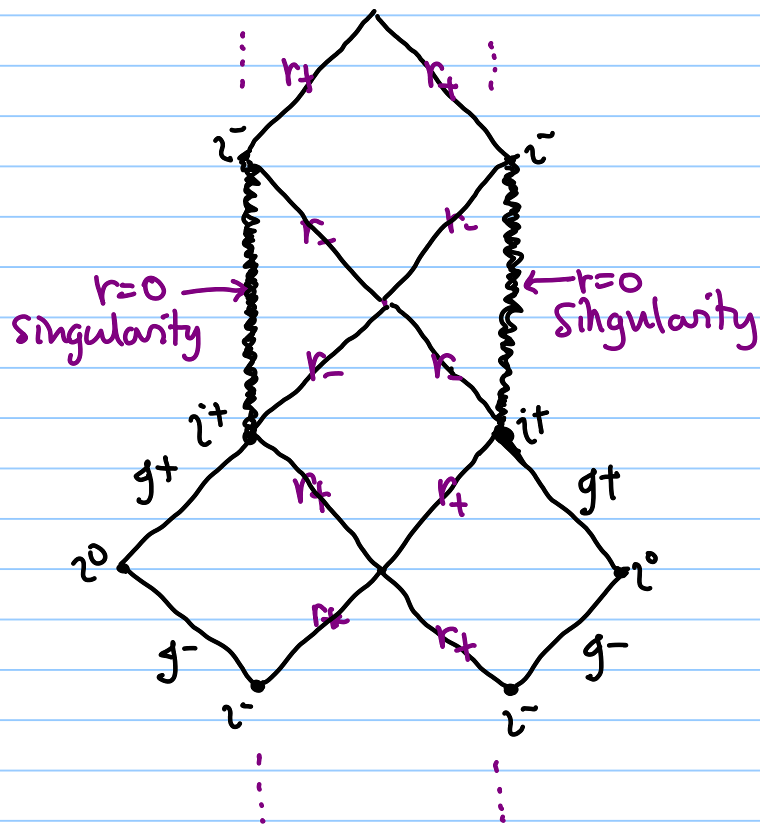
(3.2) $$\boxed{r_{ex} = GM = +\sqrt{Q^2+P^2}}$$

▷ What does the Penrose diagram look like for RN BH?

⊛ Depends on $\dfrac{M}{\sqrt{Q^2+P^2}}$

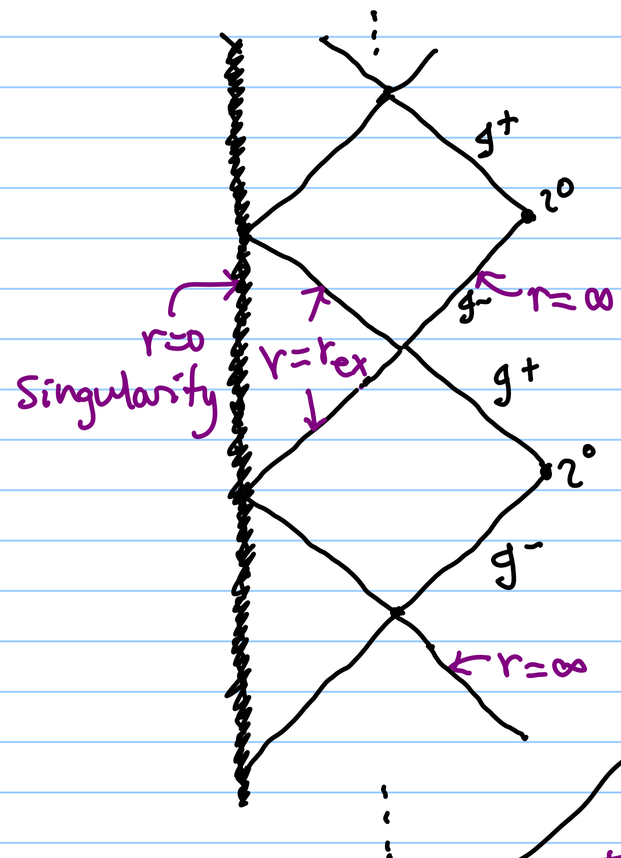
$GM > \sqrt{Q^2+P^2}$

Maximally extended geometry:-
(not relevant for gravitational collapse)




N.B.:
⊛ Research shows that the inner horizon is probably unstable
– incoming perturbations pile up there ∵ of ∞ blueshift.

$GM = \sqrt{Q^2+P^2}$ Extremal RN




- Interesting fact: when $GM = \sqrt{Q^2+P^2}$, gravity (attractive) balances EM (repulsive)

$\Rightarrow$ Supersymmetric when embedded in $\mathcal{N}=2$ SUGRA(!)

Also, can write $\rho \equiv r - r_{ex}$

$\Rightarrow$
$$ds^2 = -\frac{dt^2}{H(\rho)^2} + H(\rho)^2\left[d\rho^2 + \rho^2 d\Omega_2^2\right]$$
"Isotropic coordinates"

These are also harmonic, i.e. obey eqn $\Box x^\mu = 0$. (c.f. gravitational radiation in GR II)

- $H(\rho) = 1 + \frac{GM}{\rho}$

$\Rightarrow$ Let's consider limit as $\rho \to 0$ ($r \to r_{ex}$). We have

$$ds^2 \xrightarrow[\rho\to 0]{} -\frac{\rho^2 dt^2}{r_{ex}^2} + \frac{r_{ex}^2}{\rho^2}\left[d\rho^2 + \rho^2 d\Omega_2^2\right]$$
$$= \underbrace{-\frac{\rho^2}{r_{ex}^2}dt^2 + \frac{r_{ex}^2 d\rho^2}{\rho^2}}_{AdS_2} + \underbrace{r_{ex}^2 d\Omega_2^2}_{S^2}$$

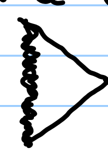
$AdS_2 \times S^2$

Link to stringy AdS/CFT duality research

$M < \sqrt{Q^2+P^2}$

For this case, • horizons walk off into the complex plane i.e. no horizon(s)!

⊛ NAKED SINGULARITY.



These are generally not preferred ∵ no predictability.

- Hope quantum gravity either rules out or resolves!

# Introduction to Black Hole Thermodynamics

No-Hair Theorem in $d=4$

$\Rightarrow$ BH have only $(M, J; Q, P)$.

horizon $\longrightarrow$ entropy, information loss

in terms of horizon area

holography

Virtual particles

Need $\hbar$, $c$ finite

pair-popping

Hawking radiation in terms of surface gravity $\kappa$

- Zeroth Law

  $\kappa =$ constant over horizon, even when $J \neq 0$

- First Law of Black Hole Mechanics

  $$dM = \left(\frac{\kappa}{2\pi}\right) \frac{dA}{4G} - \Omega\, dJ - \Phi_e\, dQ \quad \text{etc.}$$

- Second Law

  $$\frac{A}{4G} \geq 0$$

Hawking fixed $\boxed{T_H = \frac{\kappa}{2\pi}} \rightarrow \boxed{S_{BH} = \frac{A}{4G_N}}$