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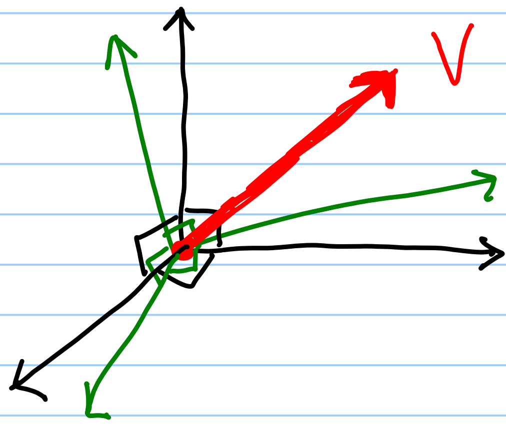
Note Title 16/09/2004

Vectors

When thinking about how vectors respond to coordinate changes, there are various ways to think about it.

We take the point of view (like in Carroll) that
the vector stays fixed
while the coordinate system changes
under the relevant transformation. (☆)



In the basis $\hat{e}_{(\mu)}$, V has components V^μ:

(1.1) $$V = V^\mu \hat{e}_{(\mu)}.$$

Under a change of coordinates $x \to x'$, the components change as

(1.2) $$V^{\mu'} = \Lambda^{\mu'}{}_{\nu} V^{\nu} \quad \text{where} \quad \Lambda^{\mu'}{}_{\nu} = \frac{\partial x^{\mu'}}{\partial x^{\nu}}$$

Since V is just a vector which sits there, V remains invariant. (☆)

(1.3) So $\hat{e}_{(\mu')} = \Lambda^{\nu}{}_{\mu'} \hat{e}_{(\nu)}$ where $(\Lambda^{\nu}{}_{\mu'})$ is the inverse matrix

how basis vectors transform.

Dual Vectors

As in linear algebra, we can define a dual basis $\hat{\theta}^{(\mu)}$ by

(2.1) $$\hat{\theta}^{(\mu)} \hat{e}_{(\nu)} = \delta^{\mu}{}_{\nu} = \begin{cases} +1, & \mu=\nu \\ 0, & \text{otherwise} \end{cases}$$

"Kronecker delta"

A dual vector ω can be written in components

(2.2) $$\omega = \omega_{\mu} \hat{\theta}^{(\mu)}$$

downstairs index

These are also called "one-forms" or "covariant vectors".

Under coord transformations, components transform:

(2.3) $$\omega_{\mu'} = \Lambda^{\nu}{}_{\mu'} \omega_{\nu}$$

The inverse matrix satisfies

$$\Lambda^{\nu}{}_{\mu'} \Lambda^{\mu'}{}_{\lambda} = \delta^{\nu}{}_{\lambda} \quad \text{and}$$

$$\Lambda^{\mu'}{}_{\nu} \Lambda^{\nu}{}_{\lambda'} = \delta^{\mu'}{}_{\lambda'} \qquad \left(\text{i.e. "}\Lambda\Lambda^{-1} = \mathbb{1} \text{ \& } \Lambda^{-1}\Lambda = \mathbb{1}\text{"}\right)$$

Math:

Dual vectors live in the cotangent space T_p^* which is a vector space.

ω's are bilinear maps $\omega: T_p \to \mathbb{R}$:

$$(a\omega_1 + b\omega_2)(V) = a\omega_1(V) + b\omega_2(V)$$

$$\omega(aV_1 + bV_2) = a\omega(V_1) + b\omega(V_2) .$$

A coordinate basis is when $\hat{\theta}^{(\mu)} = dx^\mu$;
then for some function ϕ

(3.1) $$d\phi = \frac{\partial \phi}{\partial x^\mu} dx^\mu = \frac{\partial \phi}{\partial x^\mu} \hat{\theta}^{(\mu)}$$

Then we can call the components $\frac{\partial \phi}{\partial x^\mu}$
the "gradient" dual vector

(3.2) Notation: $\frac{\partial}{\partial x^\mu} \equiv \partial_\mu$

and

(3.3) $\partial_\mu \phi \equiv \phi_{,\mu}$ ← comma sign

For any vector V and dual vector ω

(3.4) the scalar $\omega(V) = V(\omega) = V^\mu \omega_\mu$.

Notice that a vector cannot be "turned into" a scalar without the help of either
(a) something with a downstairs index (e.g. ω)
or (b) a metric tensor.

* The object with components $\eta_{\mu\nu} V^\mu$ is a dual vector(!)

Now suppose we consider something with (say) components $\omega_\mu V^\lambda$. These indices are NOT contracted (only repeated indices are summed over).
So ... is it $\delta^\mu{}_\lambda$? Nope.

It is something new – a 4x4 matrix ...

TENSORS

Think of tensors as multi-legged generalizations of vectors and dual vectors.

Mathematically, a tensor T is a multilinear map

tensor product

(4.1) $$T: \underbrace{T_p^* \otimes \cdots \otimes T_p^*}_{k \text{ times}} \otimes \underbrace{T_p \otimes \cdots \otimes T_p}_{l \text{ times}} \longrightarrow \mathbb{R}$$

It is a 'machine' with $\begin{cases} k \text{ slots for vectors} \\ l \text{ slots for dual vectors} \end{cases}$ that spits out a scalar.

T has "type" or "rank" (k,l).

Under coord transformations (viewpoint changes)

(4.2) $$T^{\mu_1' \cdots \mu_k'}{}_{\nu_1' \cdots \nu_l'} = \Lambda^{\mu_1'}{}_{\lambda_1} \cdots \Lambda^{\mu_k'}{}_{\lambda_k} \Lambda^{\sigma_1}{}_{\nu_1'} \cdots \Lambda^{\sigma_l}{}_{\nu_l'} T^{\lambda_1 \cdots \lambda_k}{}_{\sigma_1 \cdots \sigma_l}$$

Think of this as lots of matrix multiplication ☺

You can express T in terms of basis tensors

$$\hat{e}_{(\mu_1)} \otimes \cdots \otimes \hat{e}_{(\mu_k)} \otimes \hat{\theta}^{(\nu_1)} \otimes \cdots \otimes \hat{\theta}^{(\nu_l)}$$:

T's components in this basis are just $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}$.

Vectors are $(1,0)$ tensors.
Dual vectors are $(0,1)$ tensors.

Now... what use are these things? Let's set some notational conventions to help.

Manipulating tensors

- A way to reduce the number of legs on a tensor is to "contract" indices. You already know that $V^\mu \omega_\mu$ is a scalar where $\begin{cases} V^\mu \text{ are vector cpts} \\ \omega_\mu \text{ dual vector cpts.} \end{cases}$

$\underset{(1,0)}{V^\mu}\ \underset{(0,1)}{\omega_\mu}$ is a $\underset{\text{rank }(0,0)}{\text{scalar}}$

For a rank (k,ℓ) tensor T:
we can sum over a repeated index to make a rank $(k-1, \ell-1)$ tensor if one index is upstairs and one downstairs:
e.g.

(5.1) $$S^\mu{}_{\lambda\sigma} \equiv T^{\mu\nu}{}_{\lambda\nu\sigma}$$

N.B.: The ordering of indices matters!

Just like it matters whether you talk about M or M^T for a matrix M you know from linear algebra.

(5.2) In general, $T^{\mu\nu\lambda\sigma} \neq T^{\nu\mu\lambda\sigma}$, etc.

- Indices can be raised or lowered using the metric tensor $\eta_{\mu\nu}$ or its inverse $\eta^{\lambda\sigma}$.

(5.3) $$\underset{\text{rank }(0,2)}{\eta_{\mu\nu}}\ \underset{(2,0)}{\eta^{\nu\lambda}} = \underset{(1,1)}{\delta_\mu{}^\lambda}$$

All happen to be symmetric tensors
e.g. $\eta_{\mu\nu} = \eta_{\nu\mu}$
(as a matrix, $\eta = \eta^T$.)

e.g.

(5.4) $$T^\mu{}_{\lambda\sigma} = \eta^{\mu\nu} T_{\nu\lambda\sigma}$$

(5.5) and $$S_{\lambda\mu\nu} = \eta_{\mu\sigma} S_\lambda{}^\sigma{}_\nu$$

ordering matters!

Trace: for rank (n,n) tensors define the scalar

(5.6) e.g. $$\mathrm{Tr}(T) = T^\mu{}_{\nu\mu}{}^\nu$$ (contract all indices)

Symmetry and Antisymmetry

A tensor T is symmetric if, under exchange of indices, it is the same.

(6.1) e.g. $T_{\mu\nu} = T_{\nu\mu}$

(6.2) or $S_{\mu\nu\lambda} = S_{\lambda\mu\nu} = S_{\mu\lambda\nu} = S_{\lambda\mu\nu} = S_{\nu\mu\lambda} = S_{\nu\lambda\mu}$

(If it only has symmetry on k of its n indices it is not 'completely symmetric'.)

You can see that symmetry reduces the # of independent components!

Symmetrizing or anti-" } a tensor on k of its indices can be done only if all k of them are all-upstairs or all-downstairs.

Then e.g.

(6.3) $$T_{(\mu_1 \cdots \mu_k)}{}^{\lambda}{}_{\sigma} \equiv \frac{1}{k!}\left(T_{\mu_1 \cdots \mu_k}{}^{\lambda}{}_{\sigma} + T_{\mu_2 \mu_1 \cdots \mu_k}{}^{\lambda}{}_{\sigma} + \text{perms.}\right)$$

symmetrized

e.g.

(6.4) $$S_{[\mu_1 \cdots \mu_k]}{}^{\lambda\sigma} \equiv \frac{1}{k!}\left(S_{\mu_1 \cdots \mu_k}{}^{\lambda\sigma} - S_{\mu_2 \mu_1 \cdots \mu_k}{}^{\lambda\sigma} + \text{perms.}\right)$$

[alternating sign]

antisymmetrized

For a rank (2,0) or (0,2) tensor, it can always be written as the sum of a symmetric part and an antisymmetric part:

$$T_{\mu\nu} = S_{\mu\nu} + A_{\mu\nu}$$

because

$$4 \times 4 = \frac{1}{2!}(4)(4+1) + \frac{1}{2!}4(4-1)$$

In general for higher-rank tensors this sym + antisym split cannot be done: there aren't enough independent components.

Electromagnetic Field

The electric ($\vec{E}$) and magnetic ($\vec{B}$) fields can be combined into a rank two antisymmetric tensor F, whose components are $F_{\mu\nu}$!

This is plausible at a basic level, because in $d = 3+1$, a 4×4 antisymmetric tensor has
$$\frac{1}{2!}(4)(4-1) = \frac{1}{2}(4)(3) = 6$$
components - 3 for $\vec{E}$ and 3 for $\vec{B}$.

(7.1) We define $\boxed{F^{0i} = E^i}$ (antisym $\Rightarrow$ no F^{00})

and

(7.2) $\boxed{F^{ij} = \tilde{\epsilon}^{ijk} B_k}$

where
$$\tilde{\epsilon}^{ijk} = \begin{cases} +1, & (ijk) \text{ even perm. of } (123) \\ -1, & (ijk) \text{ odd perm. of } (123) \\ 0, & \text{otherwise} \end{cases}$$

e.g. $F^{01} = E^1$ (x- cpt)
$F^{12} = B_3 = B^3$ since spatial metric is $\mathbb{1}_3$.

Make a 4-vector

(7.3) $\boxed{(J^\mu) = (\rho, \vec{J})}$ $\Rightarrow$ $\boxed{\partial_\mu J^\mu = 0}$

Then the Maxwell eqns become
$$\partial_j F^{ij} - \partial_0 F^{0i} = J^i$$
$$\partial_i F^{0i} = J^0$$

(7.4) i.e. $\boxed{\partial_\nu F^{\mu\nu} = J^\mu}$ $\leftarrow$ Check

and the other eqns not involving ρ & $\vec{J}$ are

(7.5) $\boxed{\partial_{[\mu} F_{\nu\lambda]} = 0}$

Lorentz transformations are SO easy for $F_{\mu\nu}$!