Precession of Perihelion

- By Birkhoff's theorem the exterior Schwarzschild metric is very important for astrophysical applications. (Great! ☺)
- Orbits in GR actually do not follow conic sections; they are approximately ellipses that precess.
- Computed geodesic equation. Combining $r(\lambda)$ eqn and $\phi(\lambda)$ eqns gives

(1.1) $$\left(\frac{dr}{d\phi}\right)^2 + \frac{r^4}{L^2}\left(1-\frac{r_g}{r}\right) + r^2\left(1-\frac{r_g}{r}\right) = \frac{2\mathcal{E}}{L^2}r^4 \qquad \left(\mathcal{E}\equiv\tfrac{1}{2}E^2\right)$$

(1.2) Define $x \equiv \dfrac{2L^2}{r_g r}$ (useful abbreviation).

(1.3) Then $$\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{2L}{r_g}\right)^2 - 2x + x^2 - \frac{r_g^2}{2L^2}x^3 = \frac{\mathcal{E}L^2}{2r_g^2}$$

so $$\left(\frac{dx}{d\phi}\right)^2 + \left[\left(\frac{2L}{r_g}\right)^2 - \frac{\mathcal{E}L^2}{2r_g^2}\right] - 2x + x^2 - \frac{r_g^2}{2L^2}x^3 = 0$$

Acting on this with $(d/d\phi)$ gives
$$2\left(\frac{d^2x}{d\phi^2}\right)\left(\frac{dx}{d\phi}\right) - 2\left(\frac{dx}{d\phi}\right) + 2x\left(\frac{dx}{d\phi}\right) - \frac{3r_g^2}{2L^2}x^2\frac{dx}{d\phi} = 0$$

Cancelling a common factor of $2(dx/d\phi)$ (whew! ☺ cf.) gives

(1.4) $$\boxed{\frac{d^2x}{d\phi^2} - 1 + x = \boxed{3\left(\frac{r_g}{2L}\right)^2}x^2}$$ ← absent for old Newtonian story

- Expand $x = x_0 + x_1$

(1.5a) Newtonian
(1.5b) piece $$\boxed{\frac{d^2x_0}{d\phi^2} = 1 - x_0}$$

$$\boxed{\frac{d^2x_1}{d\phi^2} + x_1 = \underbrace{3\left(\frac{r_g}{2L}\right)^2 x_0^2}_{\text{"source" for } x_1}}$$ Perturbation piece

(1.6a) Solution $\boxed{x_0 = 1 + e\cos\phi}$
ellipse; $e^2 = 1-(b/a)^2$

Math (Carroll p.215) ⇒

(1.6b) $$\boxed{x_1 = 3\left(\frac{r_g}{2L}\right)^2\Big[\underbrace{\left(1+\frac{e^2}{2}\right)}_{\text{constant displacement}} + e\phi\sin\phi \underbrace{- \frac{e^2}{6}\cos 2\phi}_{\text{oscillates about } 0}\Big]}$$

Consider $x = 1 + e\cos\phi + 3\left(\frac{r_g}{2L}\right)^2 e\phi\sin\phi$ [x_0 plus 2nd term only in x_1]

$$\cong 1 + e\cos[(1-\alpha)\phi], \quad \alpha = 3(r_g/2L)^2$$

(1.7) where $$\boxed{\Delta\phi = 2\pi\alpha = \frac{6\pi G^2M^2}{L^2c^4}}$$ cf. famed precession of perihelion of Mercury: – 43″ per century ✓

(1.8) and ex x_0 $$\boxed{L^2 \cong GM(1-e^2)a}$$

RELATIVISTIC STARS — i.e. when there's stuff inside

▷ What if there is stuff in the interior of spacetime, especially if no BH forms? (Birkhoff ⇒ for spherical symmetry, do have solution = Schwarzschild exterior to region outside where $T_{\mu\nu} = 0$ ...)

no horizon @ r_g

⇔ no singularity @ $r = 0$

Minimum requirement for something to become a BH:

(2.1) $$\boxed{\lambda_c < r_g}$$

[For massive particles,] $\lambda_c = \frac{\hbar}{mc}$ whereas $r_g \sim \left(\frac{Gm}{c^2}\right)^{1/(d-3)}$

$$\Rightarrow \left(\frac{\hbar}{mc}\right)^{d-3} < \left(\frac{Gm}{c^2}\right) \quad \text{(order-of-magnitude)}$$

$$1 < G\left(\frac{mc}{\hbar}\right)^{2-d}\frac{\hbar}{c^3}$$

$$1 < \left(\frac{l_p}{\lambda_c}\right)^{d-2}$$

which is precisely (2.1) since $\frac{G\hbar}{c^3} = l_p^{d-2}$ & $[\hbar c] = M \cdot L$.

* What kind of matter is OK, and what's the equation of state?

• Let's suppose we have a perfect fluid

(2.2) $$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p\, g_{\mu\nu}$$

and try to solve Einstein equations in the star interior. [At minimum, we know that at $r = R$ the metric should match on to exterior-Schwarzschild]

Make Ansätz

(2.3) $$\boxed{ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega_2^2}$$

Choice of radial coörd such that radius$(S^2) = r$

Looking for solutions of

(2.4) $$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = T_{\mu\nu} \cdot 8\pi G$$

Need to know Einstein tensor for spacetime (2.3). We can either turn the handle by hand, or use Maple or a trusted source for general sph. sym. (2-3)'s.

Find

$$(3.1)\quad \begin{cases} G_{tt} = \frac{1}{r^2} e^{2(\alpha-\beta)} \left(2r\partial_r\beta - 1 + e^{2\beta} \right) \\ G_{rr} = \frac{1}{r^2} \left(2r\partial_r\alpha + 1 - e^{2\beta} \right) \\ G_{\theta\theta} = r^2 e^{-2\beta} \left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{1}{r}\partial_r(\alpha-\beta) \right] \\ G_{\varphi\varphi} = \sin^2\theta\, G_{\theta\theta} \end{cases}$$

For a timelike fluid in its rest frame

(3.2) $U_\mu = (e^\alpha, 0, 0, 0)$ is unit-normalized and timelike.

With (3.2), find

$$(3.3)\quad \begin{cases} T_{tt} = (e^{2\alpha})\rho \\ T_{rr} = (e^{2\beta})\, p \\ T_{\theta\theta} = (r^2)p \\ T_{\varphi\varphi} = (r^2\sin^2\theta)p \end{cases}$$

(3.4) Have $\underline{tt}$ $\quad \frac{1}{r^2} e^{-2\beta}\left[2r\partial_r\beta - 1 + e^{2\beta}\right] = 8\pi G\rho(r)$

(3.5) $\underline{rr}$ $\quad \frac{1}{r^2} e^{-2\beta}\left[2r\partial_r\alpha + 1 - e^{2\beta}\right] = 8\pi G\, p(r)$

(3.6) $\underline{\theta\theta}$ $\quad e^{-2\beta}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\,\partial_r\beta + \frac{1}{r}\partial_r(\alpha-\beta)\right] = 8\pi G p(r)$

($\underline{\varphi\varphi}$ proportional to $\theta\theta$ eqn $\because$ spherical symmetry)

Turns out: easiest way forwards is to define

$$m(r) \equiv \frac{r}{2G}\left(1 - e^{-2\beta}\right)$$

(3.7) or $\boxed{e^{2\beta} \equiv \left(1 - \frac{2Gm(r)}{r}\right)^{-1}}$

The tt equation is then

(4.1) $$\boxed{\frac{dm(r)}{dr} = 4\pi r^2 \rho(r)}$$ (sensible! ✓)

Now: if we want to integrate up to get the mass, there is a wee subtlety. Namely, that to match the interior and exterior metrics what we should do is

(4.2) write $$\bar{M} \equiv \int_0^R 4\pi (r')^2 \, d(r') \, \rho(r') \left(1 - 2\frac{Gm(r)}{r}\right)^{-1/2}$$

because this is going to match M_{ADM} for Schwarzschild. The function

(4.3) $$m(r) \overset{(4\pi)}{\equiv} \int_0^r dr' \, (r')^2 \rho(r') \quad ; \quad M \equiv m(R)$$

is the "naive" mass ; then

(4.4) $$\bar{M} - M = E_{binding}$$ ← notion that is well-defined for these stars, but not always in general in GR.

The other equation was for α

(4.5) $$\frac{d\alpha}{dr} = \left(\frac{Gm(r) + 4\pi G r^3 p(r)}{r^2 [1 - 2Gm(r)/r]}\right)$$

Can be integrated → $g_{tt} = -e^{2\alpha}$

Also, can find dp/dr via:

(4.6) $$\nabla_\mu T^{\mu\nu} = 0$$

to an equation which is simpler to work with, via

(4.7) $$\boxed{(\rho + p)\frac{d\alpha}{dr} = -\frac{dp}{dr}},$$

viz :-

(4.8) $$\boxed{\frac{dp}{dr} = -\frac{(\rho(r) + p(r))[Gm(r) + 4\pi G r^3 p(r)]}{r[r - 2Gm(r)]}}$$

Eqns (4.1) & (4.8) are together called the Tolman-Oppenheimer-Volkoff equation

or, less fancily, the eqns for hydrostatic equilibrium when Einstein gravity is your spacetime theory. ☺

(4.9) To solve, we need $\boxed{p(\rho)}$: "equation of state"

Astrophysically interesting systems often involve fluids with polytropic equations of state,
(5.1) $p = K\rho^{\gamma}$ (e.g. "$p = w\rho$" [simplest])
$\quad$ ↖ constant

See Carroll on p.234 for working out of somewhat unrealistic case of incompressible fluid.

(5.2) Take $\rho(r) = \begin{cases} \rho_*, & r \le R \\ 0, & r > R \end{cases}$

Then easy to integrate for $m(r)$:

(5.3) $m(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_*, & r \le R \\ \frac{4}{3}\pi R^3 \rho_*, & r > R \end{cases}$

Hydrostatic equilibrium eqn integrates immediately to give

(5.4) $$\boxed{\frac{p(r)}{\rho_*} = -\frac{\left(\sqrt{R^3 - r_g r^2} - R\sqrt{R - r_g}\right)}{\left(\sqrt{R^3 - r_g r^2} - 3R\sqrt{R - r_g}\right)}} \qquad r \le R$$

where

(5.5) $$r_g \equiv 2GM = \tfrac{8}{3}\pi R^3 \rho_* G$$

(Notice that the pressure goes up monotonically with decreasing radius; this makes sense because the stuff nearer the core has the weight of the stuff above it to cope with.)

$$p(0) \to \infty \quad \text{iff} \quad M > \boxed{M_{max} = \frac{4R}{9G}}$$

This is called Buchdahl's theorem.

Gravitational Collapse $\quad$ Limits

- Fermi degeneracy pressure.

White dwarf $\quad$ Neutron Star (includes pulsars) $\quad$ BLACK HOLE

¡EXTREME PHYSICS!

VERY common $\quad$ less so $\quad$ most galactic cores

Oppenheimer-Volkoff: ~3-4$M_\odot$ $\quad$ Chandrasekar: ~10$M_\odot$