Action Principle for gravity

We can derive Einstein's equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}$ from an action principle ; this actually tells us how to define $T_{\mu\nu}$ in-principle.

What I'll do is propose an action for $g_{\mu\nu}$ that gives the Einstein equation, which is consistent with decent physical principles.

First, a reminder about Classical Field Theory

In classical mechanics, our variables were

$$q^a(t)$$

coordinates ↗ ↖ non-relativistic time

and the "velocities" were

$$\dot{q}^a(t) \quad , \quad \dot{} \equiv d/dt.$$

We also defined canonical momenta

$$p_a \equiv \frac{\partial L}{\partial \dot{q}^a}$$

where the action

$$S = S[q^a(t)] = \int dt \, L(q^a, \dot{q}^b)$$

and we can form the Hamiltonian

$$H = \sum_a p_a \dot{q}^a - L$$

$$= H(q^a, p_b) \qquad \text{(lives on phase space)}$$

Hamilton's principle $\Leftrightarrow$ $\delta S = 0$

Euler-Lagrange equations

$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = 0$$

equivalent to Hamilton's equations

$$\frac{dp_a}{dt} = \{p_a, H\}$$

$$\frac{dq^a}{dt} = \{q^a, H\}$$

with $\{f, g\} = \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a} - \frac{\partial g}{\partial q^a}\frac{\partial f}{\partial p_a}$ (f, g live on phase space)

(In particular,

$$\{q^a, H\} = \frac{\partial H}{\partial p_a} \quad \text{and} \quad \{q^a, H\} = -\frac{\partial H}{\partial p_a}.$$

Field Theory: $\frac{d}{dt} \longrightarrow \partial_\mu$

and our "coordinates" are

$$q^a(t) \longrightarrow \phi^a(x^\mu)$$

Here, a is a collection of indices (could be a vector or tensor index, or an internal symmetry index (acted on by a gauge group, or something.).

The $\phi^a(x^\mu)$ are fields and in GR (i.e. physics!) they must be tensors.

Varying the action gives $\frac{\delta S[\phi^a(x^\mu)]}{\delta\phi^b} = 0$

gives, as you might expect,

$$\boxed{\frac{\partial\mathcal{L}}{\partial\phi^a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\right) = 0}$$

where D (dim of spacetime)

$$S[\phi^a(x^\mu)] = \int d^Dx\, \mathcal{L}_{matter} = \int dt\, L$$

Lag. density Lagrangian

$$= \int (d^Dx\sqrt{-g})\left(\frac{1}{\sqrt{-g}}\mathcal{L}_{matter}\right)$$

[sometimes called $(e^{-1}\mathcal{L})$ by bein-savvy people :-)]

Constructing action for $g_{\mu\nu}$

(*) What set of principles will we use to construct our action for $g_{\mu\nu}$?

#1: It should be invariant under coordinate transformations (perhaps up to a total derivative).

So if we write $S = \int d^Dx\, \mathcal{L}$ we need to make sure (a) $\int d^Dx \sqrt{-g}$ (etc.)

(b) (etc.) must be a scalar, not just a tensor.

$\Rightarrow \quad S = \int d^Dx \sqrt{-g}\,(\text{scalar})$

(*) What scalars can we build out of the metric tensor?

The easiest way I find to do this kind of thought process is by considering
- tensors first
- then contracting.

(*) What do we have to work with? We've got
$\nabla_\mu, \quad g_{\lambda\sigma}, \quad R_{\alpha\beta\gamma\delta}$.

Can't contract ∇_μ with anything to make a scalar, because it has an odd # of indices.

One option is to consider
$$g^{\alpha\beta} g_{\alpha\beta} = g^\alpha{}_\alpha = D = \text{constant}.$$

Let's call $S_\Lambda = -2(\text{const.}) \int d^Dx \sqrt{-g}\, \Lambda$

What about $\nabla^\mu \nabla^\nu g_{\mu\nu}$?

This is in-principle ok, but since we're using a metric-compatible connection so that $\nabla_\alpha g_{\mu\nu} = 0$ (which implies that $\nabla^\mu \nabla^\nu g_{\mu\nu} = 0$.)

So ... what else?

The next available thing is $R_{\alpha\beta\gamma\delta}\, g^{\alpha\gamma} g^{\beta\delta} = R$

This is the lowest-derivative-order invariant built out of ∇, g and R ... and then we would write

$$S_E = (\text{const.}) \int d^D x \sqrt{-g}\, R$$

Gives a well-defined initial-value problem because it involves only [up to] second-order derivatives of the fundamental dynamical variable $g_{\mu\nu}$.

So ... what is this (const.) ?
Being good physicists, we can work out what its dimensions must be, at least.

$g_{\mu\nu}$ itself has no dimensions because

$$\underbrace{ds^2}_{[\text{Length}]} = g_{\mu\nu} \underbrace{dx^\mu}_{[\text{Length}]} dx^\nu$$

So $\sqrt{-g}$ is dimensionless. Now, schematically,

$$R^{\cdot}{}_{\cdots} \sim \partial \Gamma^{\cdot}{}_{\cdot\cdot} + \Gamma^{\cdot}{}_{\cdot\cdot} \Gamma^{\cdot}{}_{\cdot\cdot} \quad (\text{etc.})$$

and

$$\Gamma^{\cdot}{}_{\cdot\cdot} \sim g^{\cdot\cdot}\, \partial_\cdot\, g_{\cdot\cdot} + (\text{etc.})$$

in other words, $R \sim$ 2 derivatives on g i.e.

$$[R] = \frac{1}{\text{Length}^2} \equiv \frac{1}{L^2}$$

So, what of S_{grav}?

$$\underset{\uparrow \atop \hbar}{S_{grav.}} = \underset{\uparrow}{(\text{const.})} \int \underbrace{d^D x \sqrt{-g}\, R}_{\uparrow \atop L^{D-2}}$$

Units:

so $[(\text{const.})] = L^{2-D}[\hbar]$
$= L^{2-D} \cdot M L T^{-1} L$
$= L^{4-D} M T^{-1}$

Let $(\text{const.}) = \dfrac{\hbar}{\ell_{Planck}^{D-2}} \equiv \dfrac{\hbar}{\ell_p^{D-2}}$

⊛ Is this thingy related to G_N at all?
To figure out if we're in the right ballpark, let's figure out the dimensions of G_N.

Potential for a point mass "$V = -\frac{GM}{r}$"

In D dimensions, Gauss's law ⇒ flux spreads out over $(D-2)$-sphere, rather than a 2-sphere, and the surface area for S^{D-2} goes like r^{D-2} ⇒

$$F \sim -\nabla V \sim -\frac{GMm}{r^{D-2}} \sim ma$$

i.e. $[a] = \left[\dfrac{G_N M}{r^{D-2}}\right]$

i.e. $[G] = \left[\dfrac{r^{D-2} a}{M}\right]$
$= L^{D-2}(LT^{-2}) M^{-1}$
$= L^{D-1} T^{-2} M^{-1}$

Let's compare this to

$$\left[\frac{\hbar}{\ell_p^{D-2}}\right] = L^{-D+2} MLT^{-1} L = L^{-D+4} MT^{-1}$$

So we're close; $[G^{-1}] = L^{1-D} MT^2$ ← we have right mass dimensions.

$\Rightarrow$ Try $[G^{-1} c^n] = L^{1-D} M T^2 (LT^{-1})^n$
$= L^{1+n-D} M T^{2-n}$

The dimensions match if $n = 3$

so that $[G] = \left[\frac{\ell_p^{D-2} c^3}{\hbar}\right]$

(Note: particle physicist units : $\hbar = 1 = c$.)

It is conventional to set

$$\boxed{S_{grav} = \frac{1}{16\pi G_N} \int d^D x \sqrt{-g}\,(R - 2\Lambda)}$$

and

$$S = S_{grav} + S_{matter}$$

⊕ The first step to finding the Einstein equation, which connects $\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}$ to $\underbrace{T_{\mu\nu}}$, is to compute $\frac{\delta S_{grav}}{\delta g^{\mu\nu}}$. We will later find that

$$\frac{\delta S_{grav}}{\delta g^{\mu\nu}} = -\frac{\delta S_{matter}}{\delta g^{\mu\nu}}$$

⊛ Let's vary S_{grav} w.r.t. $\underline{\underline{g^{\mu\nu}}}$ to find eqns of motion

∗ ▶ "upstairs is easier"
(c.f. upper-class British households with servants ... !)

So ... Let's do it!

Want $\delta S = \delta \int d^D x \sqrt{-g}\,(R - 2\Lambda)$ upon $\delta g^{\mu\nu}$.

How do we find $\delta\sqrt{-g}$?

- First of all : $\underset{\delta^\mu{}_\lambda}{g^\mu{}_\lambda} = g^{\mu\alpha} g_{\alpha\lambda}$ and $\delta(g^\mu{}_\lambda) = 0$ so

$$0 = \delta g^{\mu\alpha} g_{\alpha\lambda} + g^{\mu\alpha}\delta g_{\alpha\lambda}$$ ⊛

$$\Rightarrow \boxed{\delta g_{\mu\nu} = -g_{\mu\alpha} g_{\nu\beta}\,\delta g^{\alpha\beta}}$$

This relates "upstairs" and "downstairs" variations :)

So what of $\delta\sqrt{-g}$?
If we consider a matrix $M = (g_{\alpha\beta})$ and take the determinant :

$$\det M = \prod_i \lambda_i \qquad \lambda_i = \text{eigenvalues of } M$$

$$= \exp\left(\ln \prod_i \lambda_i\right)$$

$$= \exp\left(\sum_i \ln\lambda_i\right)$$

$$= \exp(\mathrm{Tr}(\ln M))$$ ← [definition of $\ln M$]

Now let's vary w.r.t. M. We have that

$$\delta(\det M) = (\det M).\ \delta(\mathrm{Tr}(\ln M))$$

i.e.

$$\frac{\delta(\det M)}{(\det M)} = \mathrm{Tr}\left(M^{-1}\delta M\right)$$ ← [cyclicity of Tr]

And here we take $\det M = (-g)$ so

$$\frac{\delta(-g)}{(-g)} = g^{\mu\nu}\delta g_{\mu\nu} = -g_{\alpha\beta}\,\delta g^{\alpha\beta}$$

so that $$\boxed{\frac{\delta\sqrt{-g}}{\sqrt{-g}} = -\frac{1}{2} g_{\alpha\beta}\,\delta g^{\alpha\beta}}$$ ⇐ not a tensor eqn ($\sqrt{-g}$ is a density)

⇒ ↖ Don't try to raise/lower indices on RHS !! – c.f. ⊛ above.

⊛ The next thing we need to find is $\frac{\delta R_{\alpha\beta}}{\delta g^{\mu\nu}}$,

This is actually not a particularly easy quantity to compute. What is easier is to find the variation of the Ricci tensor upon variation in the Christoffel connection, rather than metric(!)

The only potential fly in the ointment here is: what if the fantasy that $\delta\Gamma^{\cdot}_{\cdot\cdot}$ is independent of the $\delta g_{\cdot\cdot}$ is actually just that: fantasy ??

The super-cool observation of Palatini was that insisting on a torsion-free, metric-compatible connection does not bugger up anything ($\underset{\delta}{\nabla}$). ☺

Some authors of GR texts call this procedure "letting $\delta\Gamma$ flap in the breeze" and I like this terminology. More accurately, it's called using the "first-order formalism"; insisting $\nabla_\alpha g_{\beta\gamma} = 0$ after the fact just neatly connects $\Gamma^\alpha{}_{\beta\gamma}$ to $g_{\mu\nu}$.

OK. So let's go compute $\dfrac{\delta R^\alpha{}_{\beta\gamma\delta}}{\delta\Gamma^\mu{}_{\nu\lambda}}$.

Possible worry: $\Gamma^\mu{}_{\nu\lambda}$ is NOT a tensor (as we saw in lecture of 29 Sep, there are terms in the transformation law for $\Gamma^\mu{}_{\nu\lambda}$ which spoil tensoriality...).

Resolution: interestingly, these "nasty bits" actually go away if you take a difference of two Γ's. In particular:

(8.1) $\boxed{\delta\Gamma^\mu{}_{\nu\lambda} \text{ is a tensor.}}$ (Try it & see!)