GR 15 Sep 2004

Note Title 14/09/2004

# SPECIAL RELATIVITY



Today

§1.1 putting space and time together, the invariant interval

§1.2 spacelike, lightlike and timelike intervals and the Minkowski metric

§1.3 Lorentz transformations ("boosts") & how they differ from rotations

§1.4 What is a vector?

§1.5 and what the ☐ is a "one-form" ?!

§1.6 Tensors in Minkowski space - intro. (not scary!)

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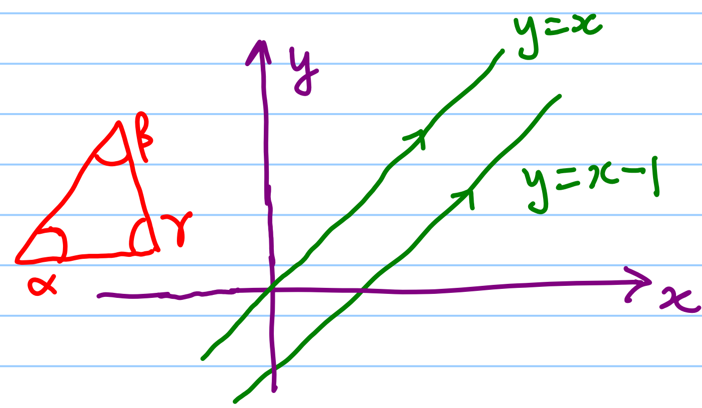
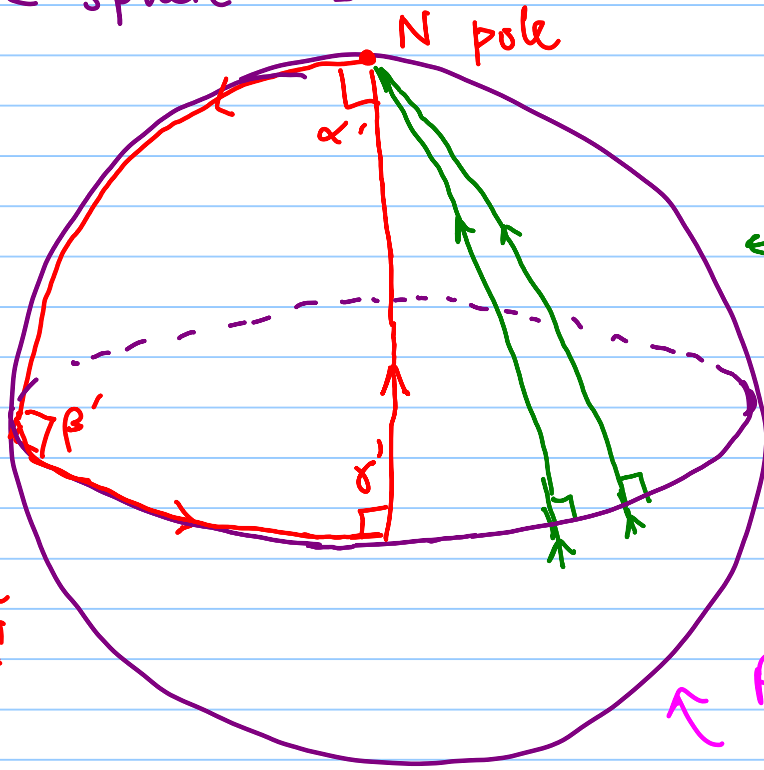
Question: do parallel lines meet & do angles in a triangle add to 180° ($\pi$)?

First, consider the plane $\mathbb{R}^2$ i.e. flat two-plane:



Now let's look at (my bad drawing of) a 2-sphere "$S^2$"




$\alpha' + \beta' + \gamma'$
$= \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2}$
$= \frac{3\pi}{2} > \pi$ !

Each of these is obviously a right angle!

← these 2 lines start out ⊥ to equator as lines of longitude but meet @ North Pole!

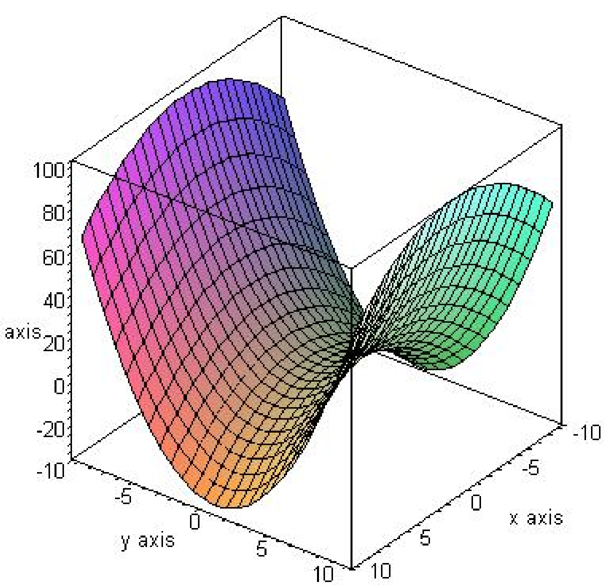
Positively Curved Space

On the other hand, on the surface of a Pringle®,

angles of a triangle sum to LESS than $\pi$

&

parallel lines DIVERGE.




← Negatively Curved Space

Let's "walk before we run" ☺

## Before curved spacetime comes flat spacetime.

This week, we assume space-time is flat. (One step at a time ☺).

* Einstein taught us that simultaneity and length depend on your reference frame.

By contrast, Newton believed that clocks everywhere all keep the same time:

$$t' = t \quad \text{or} \quad \boxed{ct' = ct}$$

and the only spatial complication was to transform

$$x' = x - vt \quad \text{or} \quad \boxed{x' = x - \left(\frac{v}{c}\right)(ct)}$$

We now know that this was just the low-speed limit of the special relativity results

$$\boxed{ct' = \gamma\left(ct - \frac{v}{c}x\right)} \quad \text{and} \quad \left(\begin{array}{l} \& \ y' = y \\ z' = z \end{array}\right)$$

$$\boxed{x' = \gamma\left(x - \frac{v}{c}\cdot ct\right)} \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Unfortunately, the law of relativistic velocity addition is pretty ugly phrased in terms of $v$'s & $\gamma$.

▷ Introduce rapidity $\zeta$ by $\boxed{\frac{v}{c} = \tanh\zeta}$

then $\gamma = \dfrac{1}{\sqrt{1 - \tanh^2\zeta}} = \dfrac{\cosh\zeta}{\sqrt{\cosh^2\zeta - \sinh^2\zeta}} = \cosh\zeta$

so that

$$ct' = (\cosh\zeta)(ct) - (\sinh\zeta)(x)$$
$$x' = -(\sinh\zeta)(ct) + (\cosh\zeta)x$$

(index "upstairs")

Let's write $(x^\mu) \equiv \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$, $\mu = 0, 1, 2, 3$ "Greek indices"

Four-vector

Then $(x^\mu)' = \underbrace{\begin{bmatrix} (\cosh\zeta) & (-\sinh\zeta) & 0 & 0 \\ (-\sinh\zeta) & (\cosh\zeta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$ $(x^\mu)$ "$(\Lambda^\mu{}_\nu)$"

This is a 4x4 matrix that looks awfully familiar ... like a rotation matrix ... except for the fact that there are hyperbolic rather than trig functions.

So let's revisit old friends the rotations, now.

For the 3-vector $(x^i) = \vec{x}$ $i = 1, 2, 3$ "Roman indices"
a rotation matrix we are used to takes

$$\vec{x} \to \vec{x}' = R\vec{x}$$

These are the matrices that are orthogonal

$$R^T R = \mathbb{1}$$

and have unit determinant ("SO(3)").

Let's rewrite this, kinda trivially:

$$R^T \mathbb{1}_3 R \overset{?}{=} \mathbb{1}_3$$

Now let's ask if $\Lambda$ satisfies anything like this.
Find that

$$\Lambda^T \mathbb{1}_4 \Lambda \neq \mathbb{1}_4 \quad .$$ ☹ ← Try it!

But if we consider

$$\eta \equiv \mathrm{diag}(-1, +1, +1, +1)$$

then

$$\boxed{\Lambda^T \eta \Lambda = \eta}$$ ☺

As a matrix, $\boxed{(\eta_{\mu\nu}) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$ it's very useful. [Good math --- = good def.]

Watch the index positioning...

This is given the fancy name "Minkowski metric".

Let's see what it does if we do with 4-vectors with $\eta$ what we usually do to find the norm of a 3-vector. With $\vec{x}$ we find $\vec{x}^T$ to get

$$\|\vec{x}\|^2 = \vec{x}^T \cdot \vec{x} = (x^1)^2 + (x^2)^2 + (x^3)^2$$
$$= \vec{x}^T \cdot \mathbb{1} \cdot \vec{x}$$

this "2" signifies the y coörd not the square of x (!)

Now let's try to mimic this :-

$$x^T \eta x = \begin{bmatrix} x^0 & x^1 & x^2 & x^3 \end{bmatrix} \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

$$= \begin{bmatrix} x^0 & x^1 & x^2 & x^3 \end{bmatrix} \begin{bmatrix} -x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

$$\boxed{x \cdot x = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2}$$

This is not positive definite (!).

In particular, the 4-vector has norm

$$x \cdot x = -(x^0)^2 + \|\vec{x}\|^2$$
$$= -c^2 t^2 + \|\vec{x}\|^2$$

A fancier way of writing this is to say

$$x \cdot x = \eta_{\mu\nu} x^\mu x^\nu$$

More generally, for 4-vectors like $x^\mu$,

$$\boxed{V \cdot W = \eta_{\mu\nu} V^\mu W^\nu \equiv \eta(V, W)}$$

where $(\eta_{\mu\nu}) \equiv \mathrm{diag}(-1, +1, +1, +1)$

So this metric takes two vectors & makes a scalar. like a '2-slot machine'.

## Interval classification

We had the norm of the 4-vector

$$x \cdot x = \eta(x,x) = x^\mu \eta_{\mu\nu} x^\nu$$

$$= -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$$

For a 4-vector $\Delta x^\mu$ denoting the space-time separation between 2 events, A & B,

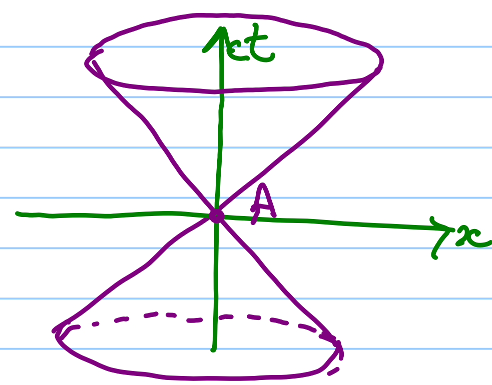
$$\Delta x \cdot \Delta x = -c^2 \Delta t^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 .$$

Suppose we simplify by letting $\Delta y = 0$ & $\Delta z = 0$.
then $\Delta x \cdot \Delta x = -c^2 \Delta t^2 + \Delta x^2$.

$\Delta x \cdot \Delta x = 0 \iff$ A & B can be connected by a light ray.

$$\left( \frac{\Delta x}{\Delta t} = \pm c \; ! \right)$$

If we put the origin at A, all points B connectible to A by a possible light ray collectively form the "light cone" for A :




For $\Delta x \cdot \Delta x > 0$
a particle connecting A & B would have to go $> c$.
This is against the law. STOP e.g. $\Delta t = 0$, $\Delta x > 0$

For $\Delta x \cdot \Delta x < 0$, connecting can be done by real particles (not imaginary ones called "tachyons".)
e.g. $\Delta t > 0$, $\Delta x = 0$

Generally, a 4-vector $V^\mu$ is classified by its norm:-

$$\eta_{\mu\nu} V^\mu V^\nu = \eta(V,V) = V \cdot V \begin{cases} = 0, & \text{lightlike} \\ < 0, & \text{timelike} \\ > 0, & \text{spacelike.} \end{cases}$$

## - Interlude -

1) Check on office hours situation:

Also: apologies - today there is an unusual exception (short-notice opportunity):

This afternoon ~4→6PM I will be "house guest" on CBC Radio One Toronto's afternoon show!

⇒ welcome to see me till 14:45 after class.

⊕ by appointment (call 978-3911)

2) Textbook update:
Carroll is SOLD OUT everywhere in N. America!
(Bookstore was not warned...)

They have rush ordered a new batch of ~25
To arrive in 2-3 (?) weeks.

⇒ I will see about photocopying 20% of it to aid HW #1 and those who don't have a copy yet

(You could also try amazon (chapters-indigo?)

* How many are we short?: $\boxed{\# = 24}$

▷ I will talk to Bookstore science buyer.

(I think)

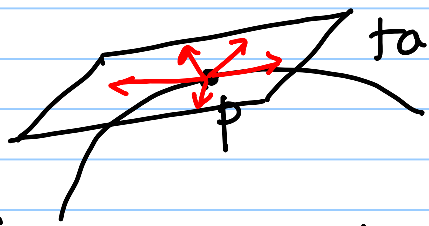
also called a "contravariant vector"

## Definition of a (4-) vector

The physical characteristic of a 3-vector that matters for us is how it acts under rotations. (By contrast, a scalar – as distinct from a mountain climber, is oblivious to rotations.)

Similarly, what matters for 4-vectors, for us, is how they transform under rotations and boosts.

A vector is also associated to a point in spacetime – its base has to stick somewhere.




made up of vectors like these

→ "tangent space" $T_p$

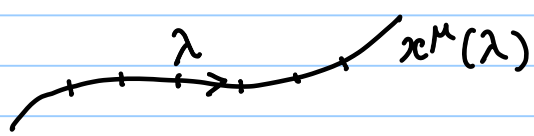
Of course, vectors satisfy the axioms you recognize from linear algebra: $(a+b)(V+W) = aV + bV + aW + bW$

Has expansion in basis vectors $\hat{e}_{(\mu)}$ [orthonormal in $\eta_{\mu\nu}$] $\boxed{W = W^\mu \hat{e}_{(\mu)}}$

Physically, when $(\Lambda^\mu{}_\nu)$ includes arbitrary $\left\{\begin{array}{ll}\text{rotations} & (3)\\ \text{boosts} & (3)\end{array}\right.$ the vector components transform as

$$\boxed{V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu} \quad (*) \quad \boxed{\Lambda^{\mu'}{}_\nu = \frac{\partial x^{\mu'}}{\partial x^\nu}}$$

A special example is when we have a path




and we make the tangent vector $V^\mu = \dfrac{dx^\mu}{d\lambda}$

(*) Has to be true regardless of the reference frame.

$$\Rightarrow \hat{e}_{(\mu')} = \Lambda^\nu{}_{\mu'} e_{(\nu)}$$

inverse matrix; transforms $x'$ to $x$