Einstein's equation – first look

We have seen that $T^{\mu\nu}$, the stress-energy tensor, is covariantly conserved:

(1.1) $$\nabla_\mu T^{\mu\nu} = 0 .$$

We have also seen that the Einstein tensor

(1.2) $$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R$$

also obeys the same equation

(1.3) $$\nabla_\mu G^{\mu\nu} = 0$$

(Tensor equations are true "upstairs or downstairs")

If we have the mantra
Matter tells spacetime how to curve &
Spacetime tells matter how to move
then it looks pretty natural to conjecture that $T^{\mu\nu}$ and $G^{\mu\nu}$ are proportional. This is what Einstein actually found! In fact,

(1.4) $$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G_N}{c^4} T_{\mu\nu}}$$ TENSOR EQUATION

This is what I like to call one of the twin pillars of 20th century physics. It is one of the most famous equations of theoretical/experimental physics.

But Einstein didn't win the Nobel for it. Δt too large.
[remark re David Gross et al ! ☺]

Pretty soon, we will work on seeing how this could come about from an action principle $(\delta S = 0)$.

It will really help if you can review Chapter 1 Section 10 of Carroll called "classical field theory". (Essentially all we do is replace
$t \to \{x^\mu\}$ and $\underbrace{\{q^a(t)\}}_{\text{"coordinates"}} \to \overbrace{\{\phi^a(x^\mu)\}}^{\text{"fields"}}$ ...)

▷ Towards Einstein's Equation

$\boxed{\text{Newtonian Gravity}}$ ("walk before we run" philosophy ☺)

Basic ideas :-

① gravity tells matter how to move

and ② matter tells gravity how strong to be.

Newton codified these two notions mathematically as

(2.1) N1) $\vec{F} = m_{\text{inertial}}\, \vec{a}$

(2.2) with $\vec{F} = m_{\text{grav.}} (-\vec{\nabla}\Phi)$ ← Newtonian potential

so when the inertial & gravitational masses are equal (which is consistent with experiment!) we have

(2.3) $$\vec{a} = -\vec{\nabla}\Phi .$$

Secondly, we have Newton's Law of Universal Gravitation :—

(2.4) N2) $\nabla^2\Phi = 4\pi G_N \rho$ ← mass density

↑ Newtonian potential

i.e. Φ is sourced by mass (Poisson eqn)

Then along came Einstein who made fundamental insights :

a) need to incorporate relativity

and b) Gravity as Space-Time.

Newtonian Limit

If GR "stands on the shoulders of Newton" then how will we recover the Newtonian limit from the "better" GR theory of gravity?

(3.1)
$$\begin{cases} \text{(i) particles move at } \left|\frac{v}{c}\right| \ll 1 & ; \\ \text{(ii) gravity is weak} & \\ \text{(iii) gravity is static } \left(\frac{\partial}{\partial t} \text{ is insignificant}\right) & \end{cases}$$

OK. So we need the 2 ideas of coupling gravity & matter (principles ① & ② on p.1)

⊛ We already know the GR version of ①: it's the geodesic equation ☺

(3.2)
$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\nu\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

where λ is affine parameter.

Let's see how this simplifies under the conditions (3.1)!

- Let's start with # (ii). If gravity is weak, it's not far off flat spacetime ⇒ we can write

(3.3)
$$\boxed{g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}}\ , \quad |h_{\mu\nu}| \ll 1$$

There's a cool trick that happens: we can construct $g^{\mu\nu}$ easily and we can, to lowest order in h, raise & lower indices with η (!)

(3.4)
$$\Rightarrow \quad g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu} \quad .$$

To see why this works, we need to check that

(4.1) $$g_{\mu\nu}\, g^{\nu\lambda} = \delta_\mu{}^\lambda \quad . \text{(definition of inverse)}$$

We've got
$$\begin{aligned} g_{\mu\nu}\, g^{\nu\lambda} &\simeq (\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\lambda} - h^{\nu\lambda}) \\ &= \eta_{\mu\nu}\eta^{\nu\lambda} + h_{\mu\nu}\eta^{\nu\lambda} - \eta_{\mu\nu} h^{\nu\lambda} + \mathcal{O}(h^2) \\ &= \delta_\mu{}^\lambda + h_\mu{}^\lambda - h_\mu{}^\lambda \\ &= \delta_\mu{}^\lambda \qquad \checkmark \text{ o.k.} \end{aligned}$$

Now, to compute $\Gamma^\mu{}_{\nu\sigma}$, we need

(4.2) $$\Gamma^\mu{}_{\nu\sigma} = \tfrac{1}{2} g^{\mu\rho}\left(g_{\nu\rho,\sigma} + g_{\sigma\rho,\nu} - g_{\nu\sigma,\rho}\right)$$
$$\simeq \tfrac{1}{2}\left(\eta^{\mu\rho} - h^{\mu\rho}\right)\left(h_{\nu\rho,\sigma} + h_{\sigma\rho,\nu} - h_{\nu\sigma,\rho}\right)$$

(4.3) $$\simeq \tfrac{1}{2}\eta^{\mu\rho}\left(h_{\nu\sigma,\rho} + h_{\sigma\rho,\nu} - h_{\nu\sigma,\rho}\right) + \mathcal{O}(h^2)$$

- Which components are turned on for Γ here? Our Newtonian condition says that

(i) $\left|\frac{dx^i}{d\tau}\right| \ll \left|\frac{dt}{d\tau}\right|$ (low velocity) so that

we need only sum over time components in the nonlinear bit in the geodesic equation: taking $\lambda = \tau$ (rather than $\lambda = a\tau + b$ ...) we've got

(4.4) $$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{00}\left(\frac{dx^0}{d\tau}\right)^2 = 0$$

So we only need to compute

(4.5) $$\Gamma^\mu{}_{00}$$, not the entire kit & caboodle

[Michele kitten story]

So what we want is

(5.1) $$\Gamma^{\mu}{}_{00} \simeq \frac{1}{2}\eta^{\mu\rho}\left(h_{0\rho,0} + h_{0\rho,0} - h_{00,\rho}\right)$$

Now we take advantage of our 3rd Newtonian condition (iii) i.e.

(5.2) $$|\partial_i h_{\mu\nu}| \gg |\partial_0 h_{\mu\nu}|$$

$$\Rightarrow \quad \Gamma^{\mu}{}_{00} \simeq \frac{1}{2}\eta^{\mu i}\left(-\partial_i h_{00}\right) + \text{(higher order terms)}$$

(5.3) $$\boxed{\Gamma^{\mu}{}_{00} \cong -\frac{1}{2}\eta^{\mu i}(\partial_i h_{00})}$$

and therefore, in the Newtonian limit,

(5.4) $$\frac{d^2x^\mu}{d\tau^2} - \frac{1}{2}\eta^{\mu i}(\partial_i h_{00})\left(\frac{dt}{d\tau}\right)^2 = 0 \qquad (\forall \mu)$$

So we've got four (D) equations:

(5.5) <u>0)</u> $$\frac{d^2t}{d\tau^2} = 0 \quad \Rightarrow \quad t = a\tau + b$$

(5.6) $$\text{so} \quad \left(\frac{dt}{d\tau}\right)^2 = a^2$$

(5.7) <u>i)</u> $$\frac{d^2x^i}{d\tau^2} = +\frac{1}{2}\partial_i h_{00}\left(\frac{dt}{d\tau}\right)^2$$

so we can change from proper time τ to lab time t :

(5.8) $$\frac{d^2x^i}{dt^2} = +\frac{1}{2}\partial_i h_{00}$$

▷ If we rename $h_{00} = -2\Phi$, then we've got

$$\vec{a} = -\vec{\nabla}\Phi$$! way cool ! ☺

(5.9) $$\Rightarrow \quad ds^2_{\text{Newtonian}} \simeq -(1+2\Phi)\,dt^2 + \delta_{ij}\,dx^i dx^j \, .$$

Tensor version of gravity field equation

(6.1) ⊛ What's the relativistic analogue of $\vec{\nabla}^2\Phi = 4\pi G_N \rho$?

- Your first guess might be to send $\vec{\nabla}^2 \to \partial^\mu \partial_\mu$ or perhaps $\nabla^\mu \nabla_\mu$, and take (say) $\rho \to T^\lambda{}_\lambda$?

The problem with this simplest modification (in the sense of William of Ockham ☺) is inconsistent with experiment !

Gravity is not described in our universe by a relativistic scalar theory.
e.g. bending of light by the Sun comes out to be only half (exactly ½) of what's measured in real life.
There are other problems too.

- Next guess might be that gravity is described by a spin-one field – a vector like A_μ of E&M.

The problem is again experiment: forces mediated by spin-1 gauge bosons can be either attractive or repulsive depending on the sign of the charges of the matter particles ... whereas gravity is always attractive, even for antimatter !

e.g. "Fall of the antiproton" experiment @ CERN ...

The above experimental results I quote are accurate to within abilities of really impressive modern experimentalists ☺

- Next simplest field we could use to describe gravity is a spin-2 tensor. Little ripples about flat space, those wee $h_{\mu\nu}$'s, are to be thought of (when quantized) as the gravitons. – "quanta of the gravitational field"

Note: can't use fermion fields to mediate (spin 1/2 or 3/2) because, to build up a big gravitational field, you need lots and lots of gravitons — whereas fermions must obey the Pauli Exclusion Principle!

- Next question: is it massless?

(7.1) Yes: $m^2_{\text{graviton}} = 0$ ← (Einstein)

(7.2) and $\boxed{c_{\text{gravity}} = c_{\text{light}}}$

There are several ways to get Einstein's equation

(7.3) $$\boxed{R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}}$$ explained next time

▷ One method (relatively sophisticated) is to begin with the Newtonian limit plus the Newtonian version of coordinate invariance and iterate with what's called the "Noether procedure".

Another is via an action principle which I will develop on Friday.

Please review Carroll §1.10