Coordinate changes – a useful "BS detector" tool ①

N.B.: Any old choice for new coordinates has to be consistent with GR principles : a well-behaved 0-form – and dx^μ must be a true d of something.

Suppose we had $\{x^\mu\}$ and wanted to switch to $\{y^\mu\}$ where

(1.1) $$y^\mu = y^\mu(x^\alpha)$$

Then

(1.2) $$dy^\mu = \frac{\partial y^\mu}{\partial x^\alpha}\, dx^\alpha \text{ , of course ;}$$

(1.3) $$\Rightarrow \underbrace{d\,(dy^\mu)}_{\underset{0}{\text{|||}}} = \frac{\partial}{\partial x^\beta}\left(\frac{\partial y^\mu}{\partial x^\alpha}\right) dx^\beta \wedge dx^\alpha + \frac{\partial y^\mu}{\partial x^\alpha}\, \underbrace{d(dx^\alpha)}_{\underset{0}{\text{|||}}}$$

(1.4) $$\Rightarrow \frac{\partial^2 y^\mu}{\partial x^\beta \partial x^\alpha} \text{ is symmetric.}$$

O.K. ... so what?

- Example

(1.5) last time: $$ds^2 = -\frac{2r_g^3}{r}\, e^{-r/2r_g}\,(d\tilde u\, d\tilde v) + r^2(\tilde u, \tilde v)\, d\Omega_2^2$$

Suppose we define

(1.6) $$\text{"}dw\text{"} \equiv \frac{2r_g^3}{\tilde r}\, e^{-r/2r_g}\, d\tilde u$$

(1.7) Then $$ds^2 = -\text{"}dw\text{"}\, d\tilde v + r^2 d\Omega_2^2$$ very nice & simple

(1.8) BUT $$d\,(\text{"}dw\text{"}) = \underbrace{\frac{\partial}{\partial r}\left(\frac{2r_g^3}{r}\, e^{-r/2r_g}\right) dr \wedge d\tilde u}_{\neq 0} + (\ldots)\, \underbrace{d^2\tilde u}_{\underset{0}{\text{|||}}}$$

inconsistent

(1.9) Note: is perfectly OK to transform $r \to w(r)$ as long as

- $w(r)$ is monotonic [good coord]
- $ddw = 0$ [trivial with $w = w(r \text{ only})$]

Penrose Diagrams

Consider (t, r) part of a spacetime i.e., suppress $\perp S^2$.
Keep it in mind; r^2 in front of $d\Omega_2^2$ describes [varying] sphere radius.

▷ Aim: portray causal structure

- (radial) Light cones are always at $45°$ on the diagram
- apply a conformal transformation $\tilde{g}_{\mu\nu} = \omega^2(x)\, g_{\mu\nu}$ to bring infinity in to finite place.

Minkowski space

(2.1) $$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2 \qquad (-\infty < t < +\infty \,,\; 0 \le r < \infty)$$

(2.2) Trajectories $t = \pm r$ are null.

Switch to null coords

(2.3) $$\begin{Bmatrix} u = t - r \\ v = t + r \end{Bmatrix}$$

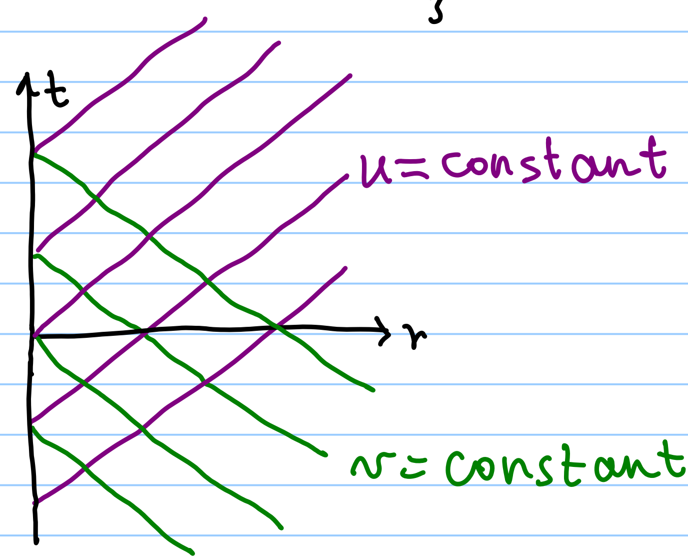
then

(2.4) $$ds^2 = -du\, dv + r^2(u, v)\, d\Omega_2^2 \qquad \left[\text{“} \tfrac{1}{2}(du\,dv + dv\,du) \text{”}\right] \text{ in Carroll...}$$

(2.5) where $\begin{Bmatrix} -\infty < u < +\infty \,, & -\infty < v < +\infty \,; \\ u \le v & \end{Bmatrix}$

On a diagram,

(2.6) →



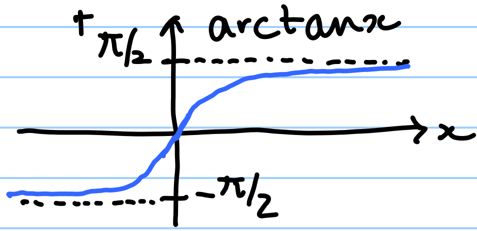
We can use a trick first met last lecture:

$$\left\{\begin{array}{l} U = \arctan(u) \\ V = \arctan(v) \end{array}\right\} \tag{3.1}$$

Brings ∞ in to a ~~finite~~ place

hence

$$-\pi/2 < U \leq +\pi/2 \,, \quad -\pi/2 < V < +\pi/2 \;; \quad U \leq V \tag{3.2}$$

(3.3) ()

(3.4) Using $g_{\mu'\nu'} = \frac{\partial x^\lambda}{\partial x^{\mu'}} \frac{\partial x^\sigma}{\partial x^{\nu'}} g_{\lambda\sigma}$, find straightforwardly that

$$ds^2 = \frac{1}{4\cos^2 U \cos^2 V}\left[-4\,dU\,dV + \sin^2(V-U)\,d\Omega_2^2\right] \tag{3.5}$$

or in

$$\left\{\begin{array}{l} T = (U+V) \\ R = (-U+V) \end{array}\right\} \tag{3.6}$$

$$\boxed{ds^2 = \frac{1}{\omega^2(T,R)}\left[-dT^2 + dR^2 + \sin^2 R\,d\Omega_2^2\right]} \tag{3.7}$$

where

$$\omega(T,R) = 2\cos\left[\tfrac{1}{2}(T-R)\right]\cos\left[\tfrac{1}{2}(T+R)\right]$$

i.e. $$\boxed{\omega(T,R) = \cos T + \cos R} \tag{3.8}$$

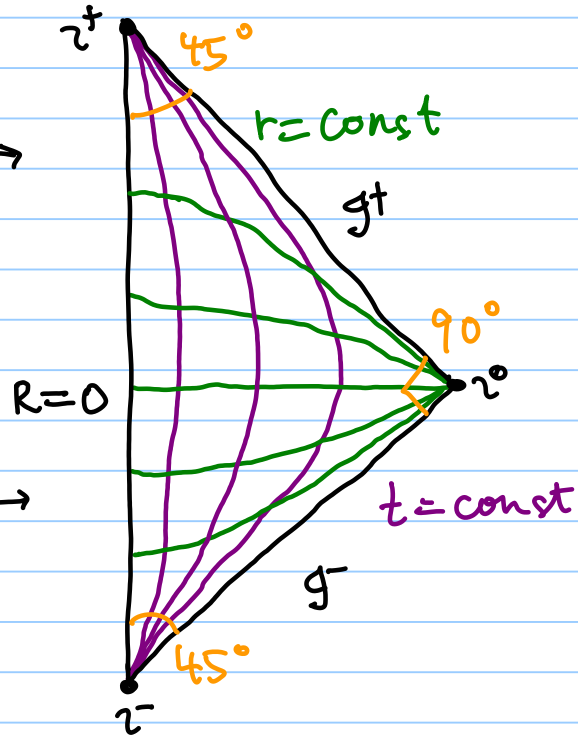
Conformal transformation of metric $\tilde{g}$ on $\mathbb{R}\times S^3$ [...].

- Curvature of S^3 not important: metric on $\mathbb{R}\times S^3$ is NOT physical; this conformally transformed metric is just auxiliary. Note also that null trajectories in radial direction are the same for $\tilde{g}$ as for g (!) $\Rightarrow$ extract causal structure from $\tilde{g}$ which is simpler. ☺

↓

Penrose Diagram of Minkowski

(4.1) →



(4.2) →

Light cone at all points: 
45°.

- i^+ = future timelike ∞
- i^- = past timelike ∞
- i^0 = spatial ∞
- $\mathscr{I}^-$ = past null ∞
- $\mathscr{I}^+$ = future null ∞

Penrose for "$a \sim t^q$ FRW"

Suppose consider

(4.3) $$ds^2 = -dt^2 + t^{2q}(dr^2 + r^2 d\Omega_2^2)$$

(4.4) or with $d\eta = t^{-q}\, dq$ (i.e. $\eta = \frac{1}{(1-q)} t^{1-q}$)

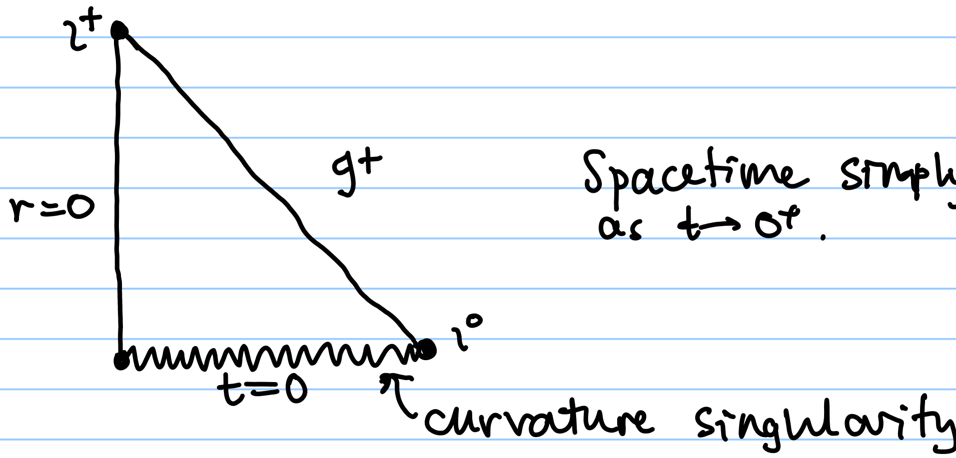
we have

(4.5) $$ds^2 = t^{2q}(-dt^2 + dr^2 + r^2 d\Omega_2^2)$$

↑ ω^{-2} times flat metric with conformal factor $t^{-q} \to \infty$ as $t \to 0$

⇒

(4.6) →

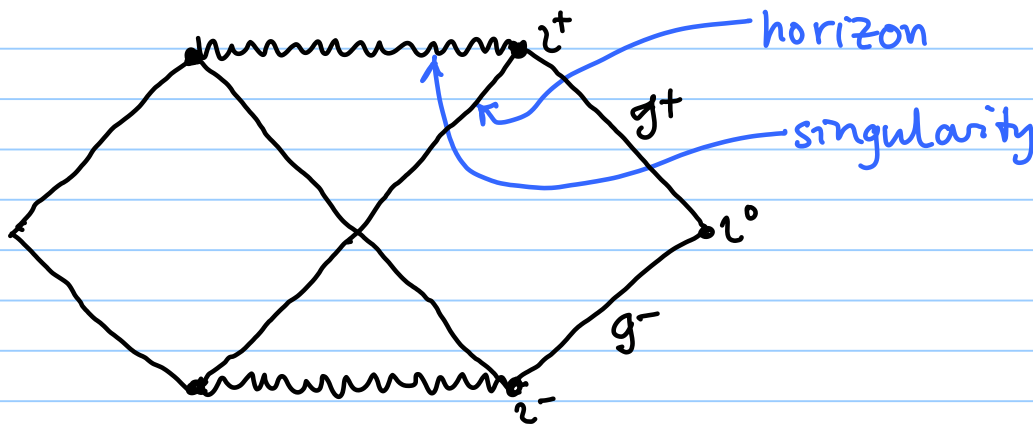


Spacetime simply <u>ends</u> as $t \to 0^+$.

Schwarzschild : eternal BH

Time-symmetric BH & WH

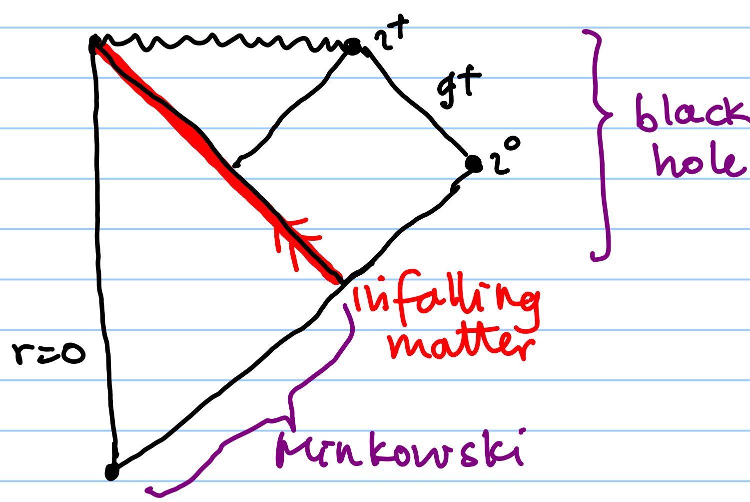
(5.1) →



Schwarzschild : gravitational collapse

Simplest collapse : null matter. Then paste on Minkowski space. ⇒

(5.2) →



Not time-symmetric - because gravitational collapse is not time-symmetric.

- Note: horizon begins after matter falls inside its own Schwarzschild radius.

(5.3) ⊛ $r_g = 2\frac{G_N M}{c^2} > l_p$ for $m > m_p \sim 10^{-5}$ g