GR 09 Nov 2004

CAUSAL STRUCTURE OF BLACK HOLES

By now, you'll be somewhat familiar with BH metric-for Schwarzschild. Let's do more to unearth the structure of this BH especially in terms of causality.

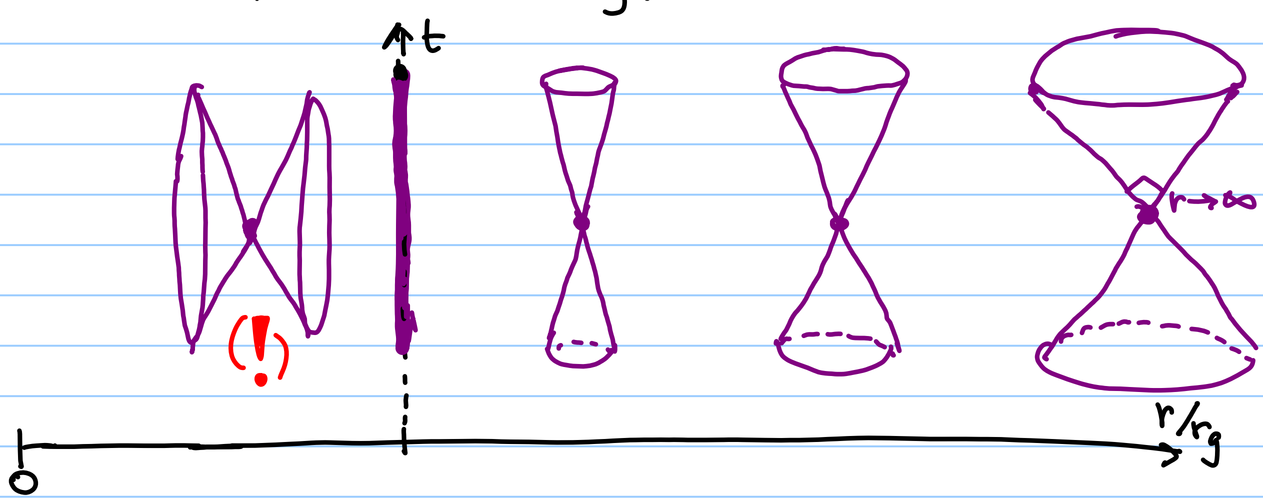
Light cones

A good place to start is the equation for a null trajectory for purely radial motion:-

(1.1) $$ds^2 = 0 = -\left(1-\frac{r_g}{r}\right)dt^2 + \left(1-\frac{r_g}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2$$

(1.2) $$\Rightarrow \quad \frac{dt}{dr} = \pm\left(1-\frac{r_g}{r}\right)^{-1} \longrightarrow \begin{cases} \pm 1 & \text{at } r \to +\infty \\ \pm\infty & \text{as } r \to r_g \end{cases}$$

So the ticks of t time in this coord system go faster and faster as $r \to r_g$!



But if the photon does not seem to Go anywhere in (r,Ω) space as $r \to r_g$, does it ever fall into a BH ?!

Actually, yes it does, but horizon-crossing does not occur in these coordinates! -because they break down at the horizon — we had nontrivial $R^\alpha{}_{\beta\gamma\delta}$ (tidal forces) even though $R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}$ is perfectly finite at $r = r_g$ (when $r_g \gg l_p$, that is!)

You wouldn't be able to discern this, though, unless you had a decent picture of the geometry and some other coordinate systems to play around with!

(2.1) We've got $\quad \dfrac{dt}{dr} = \pm \dfrac{1}{(1-r_g/r)}$

(2.2) Defining $\quad \dfrac{dt}{dr_*} = \pm 1 \quad$ gives

$$\frac{r_*}{r_g} = \frac{1}{r_g}\int \frac{dr}{(1-r_g/r)}$$

(2.3) $$\therefore \boxed{\frac{r_*}{r_g} = \frac{r}{r_g} + \ln\left(\frac{r}{r_g} - 1\right)} \qquad r \geq r_g$$

"tortoise coordinates"

Then $\quad r_* \to \infty \quad$ as $\quad r \to \infty$

and $\quad r_* \to -\infty \quad$ as $\quad r \to r_g$

(2.4) $\Rightarrow \quad r_* \in (-\infty, +\infty)$ covers the region outside r_g only.

In terms of "turtle variables" ☺

we can solve (2.1) as

(2.5) $$\boxed{t = \pm r_* + c.} \quad \text{(null radial motion)}$$

In these variables

(2.6) $$ds^2 = \left(1 - \frac{r_g}{r}\right)\left(-dt^2 + dr_*^2\right) + r^2(r_*)\, d\Omega_2^2$$

We can next try adapting our coords to null radial motion. Define

(2.7) $$\boxed{\begin{aligned} u &\equiv t - r_* \\ v &\equiv t + r_* \end{aligned}}$$

and then

Eddington–Finkelstein Coords

(2.8) $$\boxed{ds^2 = -\left(1 - \frac{r_g}{r}\right) dv^2 + \left(dv\,dr + dr\,dv\right) + r^2\, d\Omega_2^2}$$

which has det $\neq 0$ at $r = r_g$, unlike (t, r, Ω) coords.

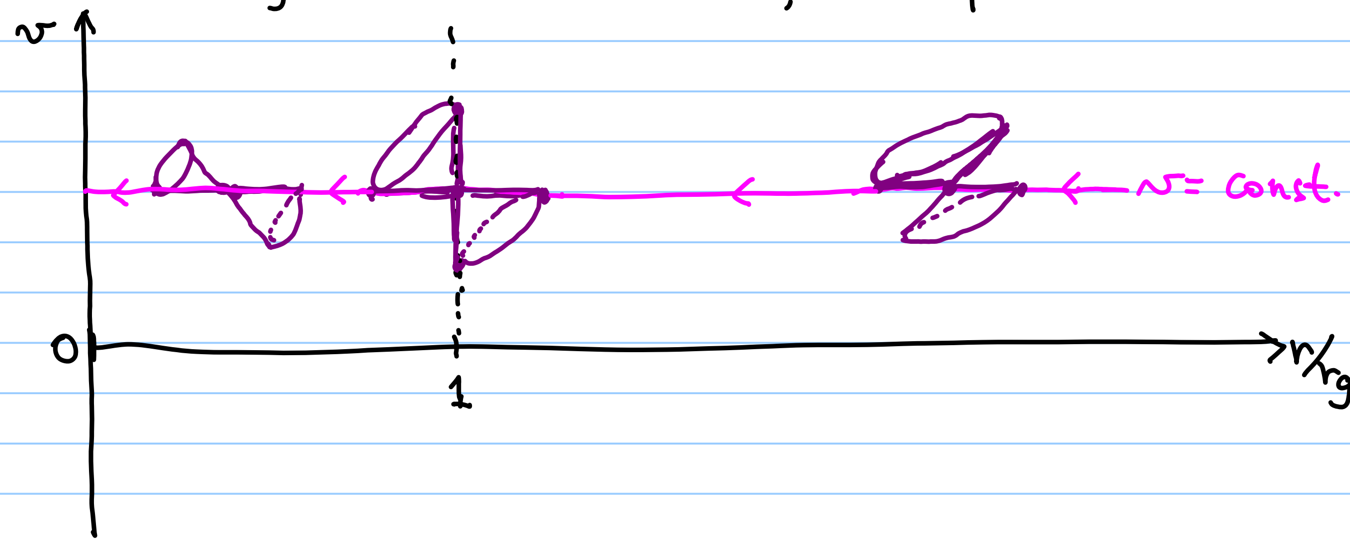
$(-g)_{EF} = -r^4 \sin^2\theta$

Then for radial null motion the equations can be written

(2.9) in terms of $\dfrac{dv}{dr} = \dfrac{dt}{dr} + \dfrac{dr_*}{dr}$; therefore

(3.1) $$\boxed{\frac{dv}{dr} = \begin{cases} 0, & \text{infalling} \\ 2\left(1-\frac{r_g}{r}\right)^{-1}, & \text{outgoing} \end{cases}}$$

In this system of coordinates, the picture is



So the light-cones in Eddington–Finkelstein coords do NOT get squished but they do turn over inside $r=r_g$.

- These E-F coords are nice, but ... are there other coords that restore the "symmetry" between u and v?

We could start by finding the coords (u, r) adapted to outgoing motion: we have

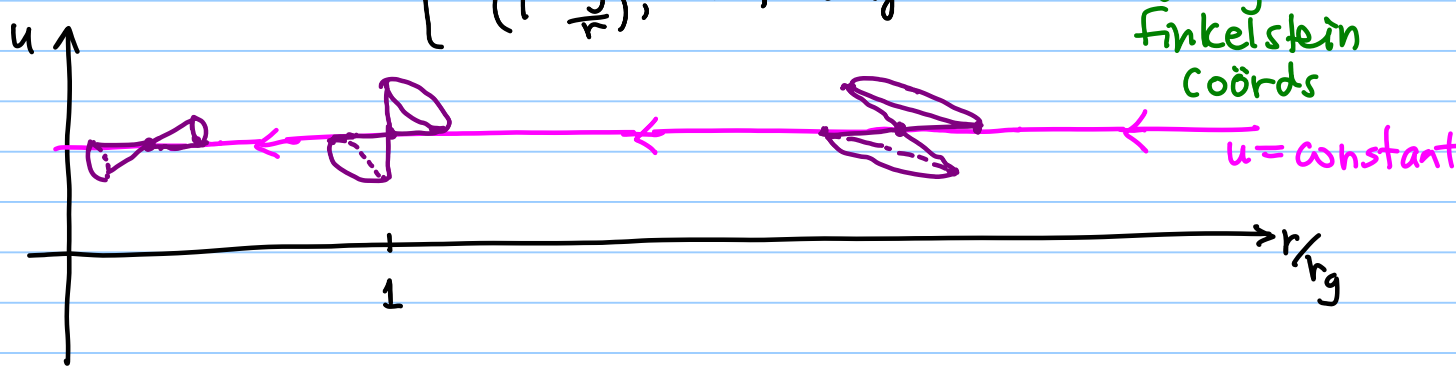
(3.2) $$\boxed{ds^2 = -\left(1-\frac{r_g}{r}\right)du^2 - (du\,dr + dr\,du) + r^2 d\Omega_2^2}$$

for which

(3.3) $$\frac{du}{dr} = \begin{cases} 0, & \text{outgoing} \\ -2\left(1-\frac{r_g}{r}\right)^{-1}, & \text{infalling} \end{cases}$$

(need to go back in time to escape BH ...)

Outgoing Eddington-Finkelstein coörds



So ... how about something like a hybrid?
Take

(4.1) $$ds^2 = -\left(1-\frac{r_g}{r}\right)\frac{1}{2}\left(du\,dv + dv\,du\right) + r^2(u,v)\,d\Omega_2^2$$

but ~~this~~ was our starting point

(4.2) where $$\frac{r}{r_g} + \ln\left(\frac{r}{r_g}-1\right) = \frac{1}{2}(v-u)$$

But we still have the problem that tortoise coords don't cover any part of spacetime for $r<r_g$ ...
so define

"Pull $r_* = -\infty$ out to finite place"

(4.3) $$\boxed{\tilde{u} = -\exp(-u/2r_g)} \qquad \boxed{\tilde{v} = +\exp(+v/2r_g)}$$

to avoid that problem. In these new Kruskal coörds we get

(4.4) $$\boxed{ds^2 = -\frac{1}{2}\left(d\tilde{u}\,d\tilde{v} + d\tilde{v}\,d\tilde{u}\right)\left[-\frac{2r_g^3}{r}e^{-r/r_g}\right] + r^2(\tilde{u},\tilde{v})\,d\Omega_2^2}$$

and we can of course make this a tad more familiar-looking by recombining

(4.5a) $$\tilde{t} = \frac{1}{2}(\tilde{u}+\tilde{v}) = \sqrt{r/r_g-1}\,e^{r/2r_g}\sinh(t/2r_g)$$

(4.5b) $$\tilde{r} = \frac{1}{2}(\tilde{u}-\tilde{v}) = \sqrt{r/r_g-1}\,e^{r/2r_g}\cosh(t/2r_g)$$

$\Rightarrow$

(4.6) $$\boxed{ds^2 = \left(-d\tilde{t}^2 + d\tilde{r}^2\right)\left(-\frac{2r_g^3}{r}e^{-r/r_g}\right) + r^2 d\Omega_2^2}$$

(4.7) with $$\boxed{\tilde{t}^2 - \tilde{r}^2 = \left(1-\frac{r}{r_g}\right)e^{r/r_g}}$$ implicitly defining $r(\tilde{t},\tilde{r})$.

Kruskal[-Szekres] coordinates

Utility

- In Kruskal coords, horizon at

(4.8) $$\tilde{t} = \pm\tilde{r}$$

- Radial null motion along

(4.9) $$\tilde{t} = \pm\tilde{r} + (\text{const.})$$

} very simple and neat :)

* In Kruskal coords, surfaces of constant r are at

(4.10) $$\tilde{t}^2 - \tilde{r}^2 = \text{constant} \qquad \text{(hyperbolae)}$$

Surfaces of constant t are at

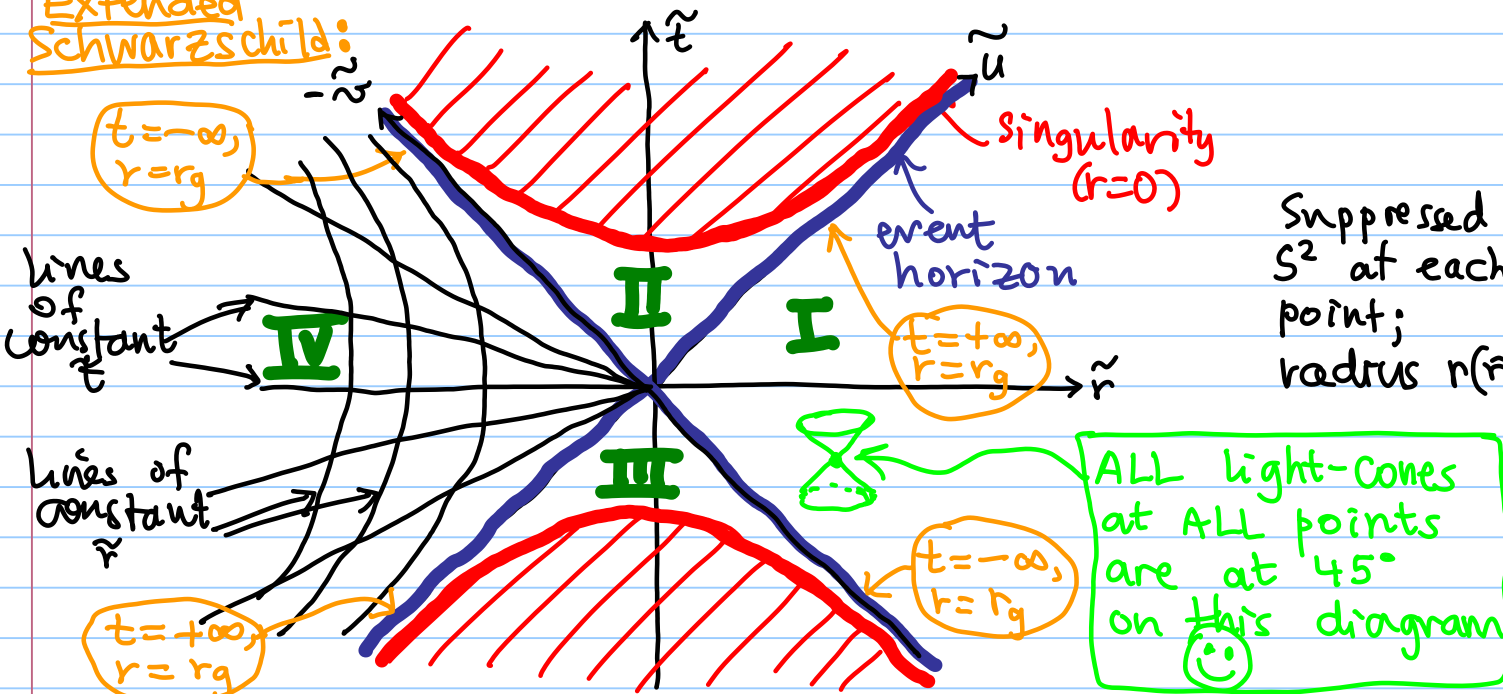
(5.1) $$\frac{\tilde{t}}{\tilde{r}} = \tanh\left(\frac{t}{2r_g}\right)$$

$\Rightarrow$ in the $(\tilde{t}, \tilde{r})$ plane, these are straight lines with slope $\tanh(r/2r_g)$.

▷ Let $(\tilde{t}, \tilde{r})$ range over all possible values aside from where the curvature singularity occurs: i.e. let

(5.2) $$\left\{ \begin{array}{l} -\infty \leq \tilde{r} \leq +\infty \\ \tilde{t}^2 - \tilde{r}^2 < 1 \end{array} \right\}$$

Maximally Extended Schwarzschild:

(5.3) 

Suppressed S^2 at each point; radius $r(\tilde{r})$

- This diagram actually has EXTRA REGIONS by comparison to the original "$r > r_g$ coordinates" (t, r, Ω)! They can be abbreviated II – IV.

- From region I we can, via future-directed null rays, go into region II. So it makes sense to interpret this part as the region behind the black hole event horizon – and you can see from the picture that the singularity ▬ is in region II.

- Suppose, from region I, we followed a past-directed null ray. Then what? According to our Kruskal diagram, we would cross a horizon to go into another region - III - with another singularity, the "mirror image" of the singularity in region II. The horizon is also a "mirror image". It is traditional to say that there is a <u>white hole</u>, the time-reverse of a black hole. This has its own horizon .

- By following future-directed null rays from III, or past-directed ones from II, see a second asymptotically flat region! But we can <u>never</u> communicate with it. (Some people talk about Schwarzschild as if it is a "wormhole" connecting asymptotically flat regions, but it isn't physical in any sense to call it a wormhole because it's not traversible: it closes up too quickly for ANY physical observer to cross from I to IV, etc. See spacelike slicings r= const. on p.228 of Carroll.)

Next time.

Conformal diagrams ("Carter-Penrose diagrams") (Appendix H)

There is a really cool idea that helps heaps when trying to picture a spacetime and especially its causal structure. Useful for grad+ research.

The basic idea is to bring infinity to a finite place so it can be drawn on the page, and to have null rays go at 45° in the (time, radius) plane. <u>Suppress</u> transverse S^2.

Start with Minkowski space-time.

Next lecture, I will show how to construct its Penrose diagram. The result will look like this →