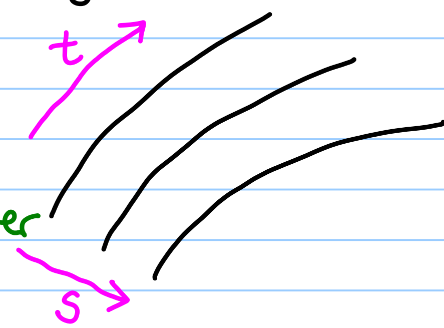
Geodesic Deviation

Suppose we have a family of geodesics that don't cross.



Tangent vector to geodesics

$$T^\mu = \frac{\partial x^\mu}{\partial t} \leftarrow \text{affine parameter}$$

Deviation vector

$$S^\mu = \frac{\partial x^\mu}{\partial s}$$

- "Relative velocity" of geodesics (definition)

$$V^\mu = (\nabla_T S)^\mu$$

(1.1) $$= T^\rho \nabla_\rho S^\mu \qquad \text{(a tensor ✓)}$$

- "Relative acceleration" of geodesics (definition)

$$A^\mu = (\nabla_T V)^\mu$$

(1.2) $$= T^\rho \nabla_\rho V^\mu = T^\rho \nabla_\rho (T^\sigma \nabla_\sigma) S^\mu$$

- S & T were carefully picked to be adapted to the above coord system:

(1.3) $$[S, T] = 0.$$

This implies

(1.4) $$S^\rho \nabla_\rho T^\mu = T^\rho \nabla_\rho S^\mu .$$

Then (algebra on Carroll p146)

(1.5) $$\boxed{A^\mu = \frac{D^2}{dt^2} S^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma}$$

This is called the geodesic deviation equation.

$\Rightarrow$ Riemann is responsible for tidal forces!

(e.g.: this makes astronaut very uncomfy crossing $M_\odot$ BH!)

Cosmological redshift

Photons move on null paths, i.e. for them $ds^2 = 0$ or $\left(\frac{ds}{d\lambda}\right)^2 = 0$. But we actually require a more restrictive condition: photons move on null geodesics.

When $\boxed{ds^2 = -dt^2 + a^2(t)\, d\vec{x}^2}$, let us pick the photon (for definiteness & w.l.o.g.) to move along $+x^1 = +x$. Then

(2.1) $$\frac{dt}{d\lambda} = a(t)\frac{dx}{d\lambda}$$

Since the only nonzero Christoffel symbols were

(2.2) $$\Gamma^0{}_{ij} = (\dot{a}a)\delta_{ij} \quad ; \quad \Gamma^i{}_{0j} = (\dot{a}/a)\,\delta^i{}_j \quad ;$$

we can easily write the geodesic equation

(2.3) $$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{d\lambda}\frac{dx^\lambda}{d\lambda} = 0$$

i.e. $$\frac{d^2t}{d\lambda^2} + (\dot{a}\,a)\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\delta_{ij} = 0 \quad \text{and}$$

$$\frac{d^2x^i}{d\lambda^2} + 2(\dot{a}/a)\frac{dt}{d\lambda}\frac{dx^i}{d\lambda} = 0$$

(2.4) $$\Rightarrow \frac{d^2t}{d\lambda^2} + \left(\frac{\dot{a}}{a}\right)\left(\frac{dt}{d\lambda}\right)^2 = 0$$ (and the same equation)

i.e. $$\frac{d}{d\lambda}\left(\frac{dt}{d\lambda}\right) = -\left(\frac{\dot{a}}{a}\frac{dt}{d\lambda}\right)\frac{dt}{d\lambda}$$

(2.5) Solved by $\frac{dt}{d\lambda} = \frac{\hbar\omega_0}{a}$ for some constant ω_0.

(2.6) For a photon, we had $\frac{dx^\mu}{d\lambda} = p^\mu$

The zeroth component is the energy

(2.7) $$\Rightarrow \boxed{E = \frac{\hbar\omega_0}{a(t)}}$$

This might seem like a funny result because $g_{00} = -1$. But the spatial bits of the metric come in because the geodesic equation (2.3-4) does depend on them!

☺ Now I want to lay some more groundwork for Einstein's equation.

A step back for a moment:

Energy-momentum tensor in special relativity

Some of you may have met T already in the study of elasticity theory or fluid mechanics. (Words like "stress", "shear", "viscosity", and so forth came up...)

Basically, $\boxed{T^{\mu\nu} = \text{flux of } p^\mu \text{ in the } x^\nu \text{ direction (S.R.)}}$

So, T^{00} is the energy density ρ (in rest frame)
and T^{0i} is the momentum density (")

T^{ij} are the stresses; in particular, diagonal components correspond to the pressure. On the other hand, off-diagonal components correspond to shearing.

$$p_i = T^{ii}$$

pressure, not momentum ↑ ↑ no sum

Perfect fluid: Need only p and ρ to specify $T^{\mu\nu}$

(3.1) i.e. $(T^{\mu\nu}) = \begin{pmatrix} \rho & & & \\ & p & & \\ & & \ddots & \\ & & & p \end{pmatrix}$

In terms of the 4-velocity u^μ of (massive!) particle(s) this is simply

(3.2) $$T^{\mu\nu}_{\text{p.f.}} = (\rho + p) u^\mu u^\nu + p \eta^{\mu\nu}$$

Dust is the case when $p=0$.

* $T^{\mu\nu}$ is typically derived from an action principle, as we will see next time. General properties? Two that must always hold:

(3.3) $T^{\mu\nu}$ is symmetric $\qquad T^{\mu\nu} = T^{\nu\mu}$

(3.4) is "covariantly" conserved: $\qquad \partial_\mu T^{\mu\nu} = 0$

The equations (3.4) in a 1+3 split become the continuity equation and the Euler equation of fluid mechanics.

Energy-momentum tensor in GR (first look) ④

A general rule which usually works ☺ is to take

$$\eta_{\mu\nu} \longrightarrow g_{\mu\nu}$$
$$\partial_\lambda \rightarrow \nabla_\lambda$$

This does work for the perfect fluid:

(4.1) $$\boxed{\nabla_\lambda T^{\lambda\mu} = 0}$$ ← Not specific for perfect fluid

where

(4.2) $$\boxed{T^{\mu\nu}_{\text{p.f.}} = (\rho+p)u^\mu u^\nu + p g^{\mu\nu}}$$ ← specific

u^μ is the 4-velocity whose norm is -1; in the rest frame its zeroth component is not 1 in general.

(Note: for a general physical system, there may be some work involved – the Belinfante procedure – to make $T^{\mu\nu}$ be symmetric. We will assume this can be done, here.)

Our cosmological metric example

Recall that our metric (for which we computed Riemann) was

(4.3) $$ds^2 = -dt^2 + a^2(t)\, d\vec{x}^2 .$$

In this metric, we can (fortunately) take

$$(u^\mu) = (1, 0, .., 0) \quad \text{for all time (!)}.$$

For a perfect fluid, to write $T^{\mu\nu}$ we need $g^{\mu\nu}$, which is

(4.4) $$(g^{\mu\nu}) = \begin{pmatrix} -1, & & \\ & a^{-2}(t) & \\ & & \ddots \; a^{-2}(t) \end{pmatrix}$$

(4.5) $$\text{so} \quad (T^{\mu\nu}) = \begin{pmatrix} \rho & & & \\ & p\,a^{-2}(t) & & \\ & & \ddots & \\ & & & p\,a^{-2}(t) \end{pmatrix}$$

We can break (4.1) down into something simpler by doing a 1+3 split, to see what the 4 conservation equations look like.

$\nabla_\mu T^{\mu\nu} = 0$ splits into

(5.1) $\nabla_\mu T^{\mu 0} = 0 = \nabla_0 T^{00} + \nabla_i T^{i0}$

and

(5.2) $\nabla_\mu T^{\mu j} = 0 = \nabla_0 T^{0j} + \nabla_i T^{ij}$

We also know that, by definition of ∇,

(5.3) $$\nabla_\mu T^{\mu\lambda} = \partial_\mu T^{\mu\lambda} + \Gamma^\mu{}_{\mu\nu} T^{\nu\lambda} + \Gamma^\lambda{}_{\mu\nu} T^{\mu\nu}$$

from Carroll (3.17)

For the time equation, we can use the Christoffel symbols I computed last time to obtain

(5.4) $$\boxed{\dot\rho = -3\frac{\dot a}{a}(p+\rho)} \qquad \left(\dot{} = \frac{\partial}{\partial t}\right)$$

needed to keep T covariantly constant (= best 'approximation' in GR to actually being constant ☺)

and

(5.5) $$\boxed{\partial_i p = 0}$$

pressure has no spatial gradient in these coords.

<u>Note</u> (for next semester) :-

(5.6) if it is possible to write $\boxed{p = w\rho}$ for your matter,

then (5.4) can be simply integrated as well as (5.4), because

$$\frac{\dot\rho}{\rho} = -3\frac{\dot a}{a}(1+w)$$

(5.7) i.e. $$\boxed{\rho \propto a^{-3(1+w)}}$$

(5.8) We say that
$$\begin{cases} \text{"matter" has } w = 0 \\ \text{"radiation" has } w = +1/3 \\ \text{"vacuum" has } w = -1 \end{cases}$$

Universe expands at different rate according to equation of state of matter in it that gravitates.

Einstein's equation: first look

We have seen that $T^{\mu\nu}$, the stress-energy tensor, is covariantly conserved:

(6.1) $$\nabla_\mu T^{\mu\nu} = 0 .$$

We have also seen that the Einstein tensor

(6.2) $$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R$$

also obeys the same equation

(6.3) $$\nabla_\mu G^{\mu\nu} = 0$$

(Tensor equations are true "upstairs or downstairs")

If we have the mantra

Matter tells spacetime how to curve &
Spacetime tells matter how to move

then it looks pretty natural to conjecture that $T^{\mu\nu}$ and $G^{\mu\nu}$ are proportional. This is what Einstein actually found! In fact,

(6.4) $$\boxed{R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \frac{8\pi G_N}{c^4} T_{\mu\nu}}$$ TENSOR EQUATION

This is what I like to call one of the twin pillars of 20th century physics. It is one of the most famous equations of theoretical/experimental physics.

But Einstein didn't win the Nobel for it. Δt too large.
[remark re David Gross et al ! ☺]

Pretty soon, we will work on seeing how this could come about from an action principle $(\delta S = 0)$.

It will really help if you can review Chapter 1 Section 10 of Carroll called "classical field theory". (Essentially all we do is replace

"fields"

$$t \to \{x^\mu\} \quad \text{and} \quad \{q^a(t)\} \to \{\phi^a(x^\mu)\} \quad \ldots)$$

"coordinates"