GR 05 Nov 2004

Note Title 31/10/2004

Last time, I introduced you to the Schwarzschild black hole (in a standard choice of coord system...), with metric:- ①

(1.1) $$\boxed{ds^2 = -\left(1-\frac{r_g}{r}\right)dt^2 + \left(1-\frac{r_g}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2}$$

(1.2) $$\boxed{r_g = 2GM/c^2}$$

We saw that $r = r_g$ is a coordinate singularity, but not a curvature singularity. That latter distinction belongs to $r=0$ alone.

Aside: GR breaks down here ⇒ need to improve theory! (of gravity)

i.e.

(1.3) $r=0$: singularity

(1.4) $r=r_g$: event horizon

Rough (and rather Newtonian) way of understanding (1.4): escape velocity for a particle $\to c$ at horizon and $> c$ inside horizon.

⇒ classically, nothing can ever escape a B.H.

☺ I will try not to use the standard joke that all experimenters falling into a BH must be [sleep-deprived, caffeinated] grad students...

Geodesics for Schwarzschild

Turning the handle for $\Gamma^\mu{}_{\lambda\sigma}$ yields

(1.5) $$\frac{d^2t}{d\lambda^2} + \frac{r_g}{r(r-r_g)}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0 \qquad ;$$

(1.6) $$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\varphi}{d\lambda}\right)^2 = 0 \qquad ;$$

(1.7) $$\frac{d^2\varphi}{d\lambda^2} + \frac{2}{r}\frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} + 2\cot\theta\frac{d\varphi}{d\lambda}\frac{d\theta}{d\lambda} = 0 \qquad ;$$

(1.8) $$\frac{d^2r}{d\lambda^2} + \frac{r_g}{2r^3}(r-r_g)\left(\frac{dt}{d\lambda}\right)^2 - \frac{r_g}{2r(r-r_g)}\left(\frac{dr}{d\lambda}\right)^2 - (r-r_g)\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{d\lambda}\right)^2\right] = 0 \quad .$$

Fortunately, the high degree of symmetry allows solving for $\frac{dx^\mu}{d\lambda}$, as follows:-

- Consider φ eqn:

(2.1) $$\left[\frac{d}{d\lambda}\left(r^2\sin^2\theta\,\frac{d\varphi}{d\lambda}\right)\right]\frac{1}{r^2\sin\theta} = \frac{d^2\varphi}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda} + 2\cot\theta\,\frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda} = 0$$

(2.2) $$\Rightarrow \boxed{r^2\sin^2\theta\,\frac{d\varphi}{d\lambda} = L_\varphi} = (\text{const.})$$

- Consider t eqn:

(2.3) $$\left(\frac{d}{d\lambda}\left[\left(1-\frac{r_g}{r}\right)^n\frac{dt}{d\lambda}\right]\right)\frac{1}{(1-r_g/r)^n} = \frac{d^2t}{d\lambda^2} + \frac{n}{(1-r_g/r)}\cdot\frac{r_g}{r^2}\frac{dr}{d\lambda}\frac{dt}{d\lambda}$$

$\Rightarrow n=1$

(2.4) $$\text{so}\quad \boxed{\left(1-\frac{r_g}{r}\right)\frac{dt}{d\lambda} = E} = (\text{const}'.) =$$

These "first integrals" are available because of SYMMETRY:

(2.5) • Stationary $\Rightarrow$ $\boxed{(K^\mu) = (1,0,0,0)}$ is a K.V.

• spherically symmetric $\Rightarrow$

(2.6) (a) can rotate till particle moves in $(x\text{-}y)$ plane i.e. @ $\theta = \pi/2$

(this step works only for 1 particle)

(2.7) (b) $\boxed{R^\mu = (0,0,0,1)}$ is a K.V.

$\uparrow$ φ direction

(2.8) In fact: $\boxed{E = -K_\mu\frac{dx^\mu}{d\lambda}}$

(2.9) and $\boxed{L = +R_\mu\frac{dx^\mu}{d\lambda}}$

Using the $\theta = \pi/2$ argument of (a) above (1-particle) we see that what remains is

(3.1) $$\frac{d^2r}{d\lambda^2} + \frac{r_g}{2r^3}(r-r_g)\left(\frac{dt}{d\lambda}\right)^2 - \frac{r_g}{2r(r-r_g)}\left(\frac{dr}{d\lambda}\right)^2 - (r-r_g)\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{d\lambda}\right)^2\right] = 0$$

and now

$$[\ldots] = 0 + \left(\frac{d\varphi}{d\lambda}\right)^2 = \frac{L^2}{r^4}$$

so that

(3.2) $$\frac{d^2r}{d\lambda^2} + \frac{r_g}{2r^2}\left(1-\frac{r_g}{r}\right)^{-1}E^2 - \frac{r_g}{2r^2}\left(1-\frac{r_g}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - \frac{L^2}{r^3}\left(1-\frac{r_g}{r}\right) = 0$$

This actually has a first integral too, most easily computed by realizing that

(3.3) $$\boxed{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = -\epsilon}$$

must be constant along a geodesic (we varied $\sqrt{\ }$ of this to get the geodesic eqn! :-))

For us,

$$-\epsilon = -\left(1-\frac{r_g}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{r_g}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\varphi}{d\lambda}\right)^2$$

$$\epsilon = \left(1-\frac{r_g}{r}\right)^{-1}E^2 - \left(1-\frac{r_g}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - \frac{L^2}{r^2}$$

i.e. $$\left(\frac{dr}{d\lambda}\right)^2 = -\left(\epsilon + \frac{L^2}{r^2}\right)\left(1-\frac{r_g}{r}\right) + E^2 \qquad (\phi)$$

(3.4) $$\Rightarrow \boxed{\frac{dr}{d\lambda} = \underbrace{\mp}_{\text{infalling/outgoing geodesic}}\sqrt{E^2 - \left(1-\frac{r_g}{r}\right)\left(\epsilon+\frac{L^2}{r^2}\right)}}$$

▷ Another way to think about this equation is to notice that (ϕ) can be rewritten

(3.5) $$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V_{eff}(r) = \left(\frac{E^2}{2}\right)$$

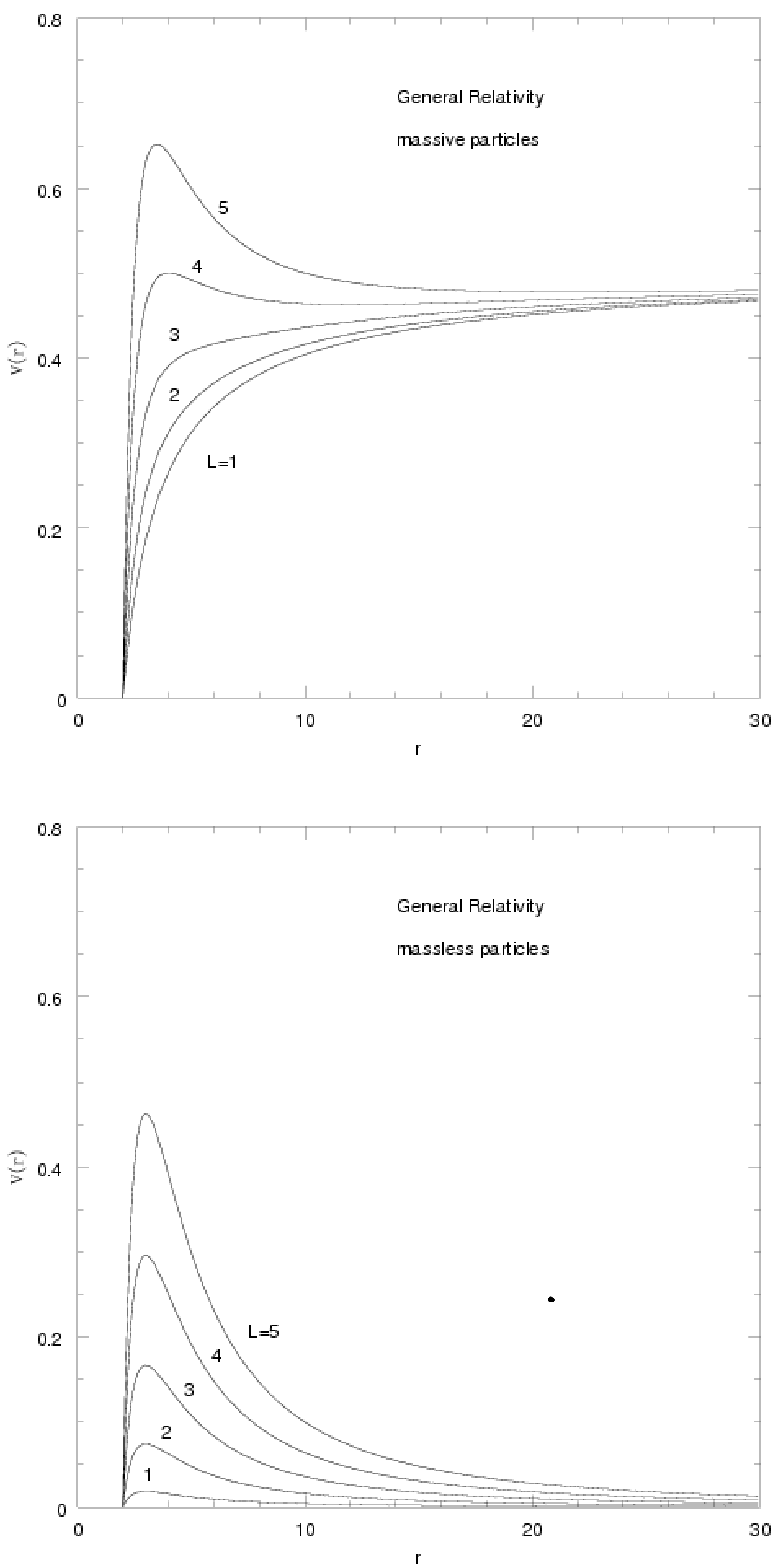
(like K.E. + P.E. = total energy = conserved)

⇒ $r(\lambda)$ obeys equation of non-relativistic fame where we imagine λ is non-rel time and

(3.6) $$\boxed{V_{eff} = -\left(1-\frac{r_g}{r}\right)\left(\epsilon+\frac{L^2}{r^2}\right)} = -\epsilon - \frac{L^2}{r^2} + \frac{\epsilon r_g}{r} + \frac{L^2 r_g}{r^3}$$

④

Picturing $V_{eff}(r(\lambda))$



(*) Please Do ask questions; this is crucial stuff! ☺

Discussion of the physics

The question of main interest is whether there are any turning points for the motion; this will tell us whether

- the probe ("test") particle falls in, or
- the probe can do a circular orbit, or
- the probe misses & goes out to $r \to \infty$ again.

Circular orbit

(5.1) For this, we require $\frac{dV_{eff}}{dr} = 0$

(5.2) Now, $\frac{dV_{eff}}{dr} = \frac{2L^2}{r^3} - \frac{\epsilon r_g}{r^2} - \frac{3L^2 r_g}{r^4}$

$= 0$ at $r = r_c$

$$\Rightarrow \epsilon r_g r_c^2 - 2L^2 r_c + 3L^2 r_g = 0$$

So $r_c = \left(+2L^2 \pm \sqrt{4L^4 - 12L^2 \epsilon r_g^2}\right)\frac{1}{2\epsilon r_g}$

(5.3) $= \frac{L^2}{\epsilon r_g} \pm \frac{L^2}{\epsilon r_g}\sqrt{1 - 3\frac{\epsilon r_g^2}{L^2}}$ when $\epsilon \neq 0$ (*)

and when $\epsilon = 0$ we have

(5.4) $\boxed{r_{c(ICO)} = \frac{3}{2} r_g = 3GM}$ ← Innermost circular orbit

Expanding (*) for small-ϵ we have

$r_c \simeq \frac{L^2}{\epsilon r_g}\left[1 \pm \left(1 - \frac{3}{2}\frac{\epsilon r_g^2}{L^2}\right) + \mathcal{O}(\epsilon^2)\right]$

(5.6) $= \frac{L^2}{\epsilon r_g}(1 \pm 1) \mp \frac{3}{2} r_g$ ⇒ ⊖ sqrt sign choice

(5.7) $\Rightarrow \boxed{r_c = \frac{L^2}{\epsilon r_g}\left[1 \mp \sqrt{1 - 3\frac{\epsilon r_g^2}{L^2}}\right]}$ (small-ish L^2) (‡)

larger L^2

For $m^2 > 0$ & $L^2 \gg 1$, there are actually two solutions which lie outside r_{c*}, given by

(5.8) $r_{c,1} \simeq \frac{L^2}{GM}$ and $r_{c,2} \simeq 3GM^+$

(6.1) N.B.: $\epsilon = 0$ for massless particles
$\Rightarrow$ photons can orbit forever at $r_c = r_{c*}$ (ONLY!)
Any photon moving a bit in or out must either fall into the black hole or escape to ∞.
It may buzz around the BH (outside of $r=r_g$) a few times before flitting off to $r \to \infty$ ☺

(6.2) (‡) In fact, for $L^2 \gg 1$ this implies that the inner orbit is the unstable one, as it matches with the photonic circular orbit.
$\Rightarrow$ ⊛ the stable orbit is at larger radius! ($L^2 \gg 1$)

- When do the stable & unstable orbits coalesce?

(6.3) When $\sqrt{1 - \frac{3\epsilon r_g^2}{L^2}} = 0$

(N.B.: impossible for photons)

(6.4) $\Rightarrow$ $L^2 = 3\epsilon r_g^2$ so that

(6.5) $$\boxed{r_{c,\,ISCO} = 3r_g} = 6GM = 2r_{c,\,ICO}$$

(P.S. All of this is analyzed in GR.
If there were an "alternative" theory of gravity, perhaps involving a scalar-tensor story and/or other terms in $\mathcal{L}$, and/or we had different coupling of $g_{\mu\nu}$ to matter fields than we do, then these calculations would need to be redone.)