GR 03 dec 2004

Note Title 02/12/2004

Quintessential Physics question: how well do we know what we think we know?

And so it is with extra dimensions of space. These are predicted by string theory.

(Note: extra dimensions of time are very hard to fathom, except classically. e.g. which time would we pick to form $H = -i\hbar\,\partial/\partial t$?)

If one extra dimension is on S^1:

Experiment: $\begin{cases} \text{if we're allowed in, size} < 10^{-18}\,\text{cm} \\ \text{if only gravity is allowed in, size} < 50\,\mu\text{m (!)} \end{cases}$

Dimensionful constants

Identify G_D from $S_D \supset \frac{1}{16\pi G_D}\int d^Dx\,\sqrt{-g_D}\,R_D$

If fields in problem don't depend on some $\tilde{d}$ coords, then $S_{d=D-\tilde{d}} \supset \frac{1}{16\pi G_D}\left(\int d^{\tilde{d}}x\right)\int d^dx\,\left(\sqrt{-g_d}\,R_d + \dots\right)$

$$\Rightarrow \quad G_d = \frac{G_D}{V_{\tilde{d}}} \qquad\qquad l_{p,d}^{\,d-2} = \frac{l_{p,D}^{\,D-2}}{V_{\tilde{d}}} = \frac{l_{p,D}^{\,\tilde{d}}\, l_{p,D}^{\,d-2}}{V_d}$$

Defining $16\pi G_D \equiv (2\pi)^{D-3}\, l_{p,D}^{\,D-2}$ gives

$$\left(\frac{l_{p,D}}{l_{p,d}}\right)^{d-2} = \frac{V_{\tilde{d}}}{(2\pi\, l_{p,D})^{\tilde{d}}}$$

So to have $l_{P,D} \gg l_{P,d}$ (bigger Planck length in higher-d) need $V_{\tilde{d}} \gg (2\pi l_{P,D})^{\tilde{d}}$ i.e. large extra dimensions!

Example : $D=6$, $d=4$ $\Rightarrow$ $\left(\frac{l_{P,6}}{l_{P,4}}\right)^2 = \frac{V_2}{(2\pi l_{P,6})^2}$

Let $V_2 \equiv (2\pi R)^2$.
Then $l_{P,6} = \sqrt{R\, l_{P,4}}$

Dimensional reduction (Kaluza-Klein)

- Suppose have field $\phi(x^\mu, y^\alpha)$
  lower-d spacetime (x^μ)   extra dimensions (y^α)

If $\{y^\alpha\}$ curled up (compactified) say on Torus, then expand $\phi(x^\mu, y^\alpha) = \sum_{n,\alpha} \phi_n(x^\mu)\, e^{i k_\alpha y^\alpha}$, $k_\alpha = \frac{2\pi n_\alpha}{(L_\alpha = 2\pi R_\alpha)}$

e.g. $\phi(x^\mu, y) = \sum_n \phi_n(x^\mu)\, e^{iky}$, $k = \frac{n}{R}$

- If $\Box_D \phi = 0$, then what happens to e.o.m in $d<D$?

$$\Box_D \phi = \left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial y^\alpha}\frac{\partial}{\partial y_\alpha}\right)\phi(x^\mu, y^\alpha)$$
$$= \sum_n \left(\partial^\mu\partial_\mu + (-1)\, k^\alpha k_\alpha\right)\phi_n(x^\mu), \quad k_\alpha = \frac{n_\alpha}{R_\alpha}$$
$$\Rightarrow \quad \Box_d \phi_n = \underbrace{k^\alpha k_\alpha}_{\text{this is } m_d^2\,!}\phi_n$$

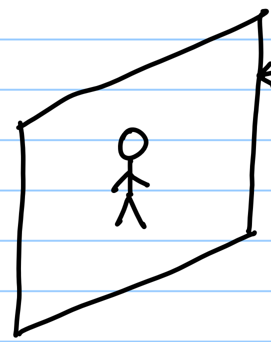
For y on circle S^1 $\therefore$ have
$$|\vec{p}|^2 = |\hbar \vec{k}|^2 = \left|\frac{\hbar n}{R}\right|^2 = \left|\frac{hn}{2\pi R}\right|^2$$

and hence $\boxed{m_n = \frac{1}{R}\hbar c\, n}$ quantized

it costs energy to excite field oscillations in T^d part.
Quantum physics : can't excite arbitrarily weakly!

Two types of extra dimensions discussed lately
(a) "ADD" : flat
(b) "RS" : warped

Important idea (imported directly from string theory)
= branes.



brane
dimension $d = p+1$

c.f.

bulk has D dimensions

Kaluza-Klein

Suppose have $g_{\mu\nu}$ in higher dimension, D.
Locally has form of Minkowski $\mathbb{R}^{1,D-1}$ which has symmetry group $ISO(1, D-1)$. Breaking this up into $D = (\tilde{d}) + (d)$ gives $ISO(1, d-1)$ ... and we will have

$$g_{\mu\nu} \to \hat{g}_{\mu\nu}, \; A_{\mu\alpha}, \; M_{\alpha\beta}$$

vectors ($A_{\mu\alpha}$), scalars ($M_{\alpha\beta}$)

• For

$$ds^2 = e^{2\alpha\hat{\chi}} d\hat{s}^2 + e^{2\beta\hat{\chi}} \left(dz + \hat{A}_{\hat{\mu}} dx^{\hat{\mu}} \right)^2 ,$$

with $\beta = (2 - D)\alpha$ and $\alpha^2 = 1/[2(D-1)(D-2)]$

$$\sqrt{-g} R_g = \sqrt{-\hat{g}} \left(R_{\hat{g}} - \tfrac{1}{2} (\partial \hat{\chi})^2 - \tfrac{1}{4} e^{-2(D-1)\alpha\hat{\chi}} F^2 \right) ,$$

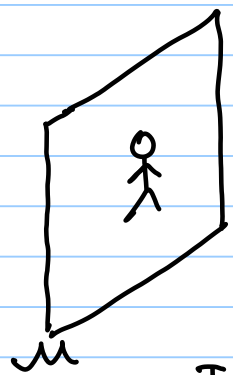
where F is the field strength of $\hat{A}$.

Einstein tried to unify gravity and E&M this way

Naturally get $U(1)$ gauge fields & (unwanted!) $m^2 = 0$ scalars.

Randall–Sundrum (brane: Codimension one)

Brane causes warping of space-time



$$S_{matter} = \lambda \int_M d^4x \sqrt{-g_4}$$

$$S_{gravity} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g_5}\, R_5$$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}} \leftarrow \text{(5-dim. metric!)}$$

Assume metric factorizes

$$ds^2 = e^{2A(z)} dx^\mu dx^\nu g_{\mu\nu} + dz^2$$

and solve e-o-m

$$\Rightarrow \quad e^{2A(z)} = e^{-2k|z|}$$

k related to G_5 & λ

fails to be differentiable at $z=0$; this is location of δ-function brane matter source.

$T_{\mu\nu} \propto \lambda g_{\mu\nu} \Rightarrow$ pressure $p = w\rho$ with $w = -1$

- More generally:
  Can have various geometries for brane & bulk, but e-o-m demand tight relationship holds.
  Also, need to worry: will brane location change with time? (brane moves).

Much recent work, on cosmological implications of branes. (thousands of papers...)