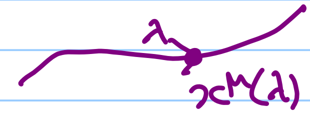
GR 01 Oct 04

Note Title 30/09/2004

Geodesic equation

Last time, we saw that a geodesic is a path that parallel-transports its own tangent vector:

(1.1) $$\boxed{\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\nu\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0}$$



This is a very important equation!

We can get another perspective on geodesics from a variational principle. Consider a massive particle, and its proper time along its path $x^\mu(\lambda)$:-

$$\Delta\tau = \int d\tau \sqrt{-\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}g_{\mu\nu}}$$

When $x^\mu \to x^\mu + \delta x^\mu$,

$$g_{\mu\nu} \to g_{\mu\nu} + (\partial_\sigma g_{\mu\nu})\,\delta x^\sigma .$$

Varying, we have

$$\delta\tau = \int d\tau \frac{1}{2\sqrt{\cdots}}\,\delta\left(-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)$$

$$= \int d\tau \frac{1}{2\sqrt{\cdots}}\left[(\partial_\sigma g_{\mu\nu})\,\delta x^\sigma \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\left(\frac{d}{d\tau}\delta x^\mu\right)\frac{dx^\nu}{d\tau} + (\mu\leftrightarrow\nu)\right]$$

$$= \int d\tau \frac{1}{2\sqrt{\cdots}}\left[(\partial_\sigma g_{\mu\nu})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\sigma - (\partial_\sigma g_{\mu\nu})\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\mu + (\mu\leftrightarrow\nu) - g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}\delta x^\mu - (\mu\leftrightarrow\nu)\right]$$

$$= \int d\tau \frac{1}{\sqrt{\cdots}}\left[-g_{\mu\sigma}\frac{d^2x^\mu}{d\tau^2} + g_{\mu\sigma}\Gamma^\sigma{}_{\nu\alpha}\frac{dx^\nu}{d\tau}\frac{dx^\alpha}{d\tau}\right]\delta x^\sigma$$

$= 0 \Rightarrow$ geodesic equation (1.1) ☺
with τ as parameter

Affine parameter

This is defined to be $\lambda = a\tau + b$ for constants a, b i.e. a λ is an affine parameter if it is linearly related to τ.* Our geodesic equation (1.1) requires just such an affine parametrization (!)

(* for a massive particle.)

For a massless particle, we just require that the geodesic equation is satisfied, to pick out a λ which is an affine parameter.

For either $m^2 > 0$ or $m^2 = 0$ particles, the geodesic equation can be written as

(2.1) $$\boxed{p^\lambda \nabla_\lambda p^\mu = 0}$$

For the $m^2 > 0$ particle, we use $p^\mu = m u^\mu = m \frac{dx^\mu}{d\tau}$ in this equation.

Question : If a geodesic extremizes proper time for a massive particle, does it minimize τ or does it maximize it?!

Answer : maximizes (!)
Reason? If we were to lower $\Delta\tau$ along a changed path, we would get closer to $\Delta\tau = 0$ which is a null path; to go lower to $\Delta\tau < 0$, we'd have to use an illegal spacelike path. So minimizing $\Delta\tau$ doesn't make sense; in fact, we maximize it via variational principle.

This is connected to the "twin paradox": the twin who stays on a geodesic (home) experiences more proper time than the astronaut sister who flies around the galaxy accelerating all over the place ;-)

Geodesic completeness

If all geodesics on a spacetime manifold go "as far as they please", it's a geodesically complete manifold. But if some geodesic(s) bang into a singularity – or end prematurely – then it's geodesically incomplete. (For spacetimes with matter this is the generic case, actually.)

An example

Let's look at spatially homogeneous metrics of the form

(3.1) $$ds^2 = -c^2dt^2 + a^2(t)\, d\vec{x}^{\,2}$$

If we want to find the geodesics in this set of coords, we need to find $\Gamma^\mu{}_{\alpha\beta}$. We have

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)$$

In our metric, the only nonzero derivatives are time derivatives of spatial parts of the metric, and the metric is diagonal.

$$\Rightarrow \quad \Gamma^0{}_{00} = 0 \qquad \Gamma^i{}_{00} = 0$$
$$\Gamma^0{}_{0i} = 0 \qquad \Gamma^i{}_{0j} \neq 0$$
$$\Gamma^0{}_{ij} \neq 0 \qquad \Gamma^i{}_{jk} = 0$$

Let's find the nonzero ones.

$c=1$

$$\Gamma^0{}_{ij} = \frac{1}{2} g^{00}\left(\partial_i g_{0j} + \partial_j g_{0i} - \partial_0 g_{ij}\right) = -\frac{1}{2} g^{00}\partial_0 g_{ij}$$

$$= -\frac{1}{2}(-1)\, 2a\dot{a}\,\delta_{ij} \quad \text{where} \quad \cdot \equiv \frac{d}{dt}$$

(3.2) $$\boxed{\Gamma^0{}_{ij} = a\dot{a}\,\delta_{ij}}$$

$$\Gamma^i{}_{0j} = \frac{1}{2} g^{ik}\left(\partial_0 g_{kj} + \partial_j g_{k0} - \partial_k g_{0j}\right) = \frac{1}{2} g^{ik}\partial_0 g_{kj}$$

$$= \frac{1}{2} a^{-2}\,\delta_{kj}\,(2a\dot{a})\,\delta^{ik}$$

(3.3) $$\boxed{\Gamma^i{}_{0j} = \frac{\dot{a}}{a}\,\delta^i{}_j}$$

The geodesics are given by

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \qquad \text{for massive particles}$$

i.e. (*)
$$\begin{cases} \dfrac{d^2t}{d\tau^2} + a\dot{a}\,\dfrac{dx^i}{d\tau}\dfrac{dx^j}{d\tau}\delta_{ij} = 0 \\[2ex] \dfrac{d^2x^i}{d\tau^2} + \left(\dfrac{\dot{a}}{a}\right)\dfrac{dt}{d\tau}\dfrac{dx^i}{d\tau} = 0 \end{cases}$$

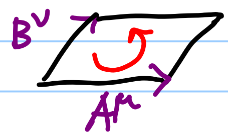
For null geodesics, $\quad \dfrac{c\,dt}{d\lambda} = \pm a\left|\dfrac{d\vec{x}}{d\lambda}\right| \quad \ldots$

(*) In terms of velocity & acceleration (1st & 2nd deriv's of x^μ).

Curvature

- For our S^2 example last time, we discovered that parallel transporting a vector around a closed loop does not necessarily return it to its original state.
  * The extent to which this fails is called curvature.

Equations? Consider an infinitesimal loop



non-zero area if A not // to B.

Under parallel-transport around the loop we get a change in V, δV, which is also a vector; in order to express δV in terms of V, A, B we need a $(1,3)$ tensor:

$$\delta V^\mu = R^\mu{}_{\nu\alpha\beta} V^\nu A^\alpha B^\beta$$

- Another way to define curvature is via the commutator of covariant derivatives:

(4.1) $$\boxed{[\nabla_\mu, \nabla_\nu] V^\alpha = R^\alpha{}_{\beta\mu\nu} V^\beta}$$ (for a torsion-free connection)

It can be written in terms of "$\partial\Gamma$" and "$\Gamma\times\Gamma$":

(4.2) $$\boxed{R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha{}_{\nu\beta} - \partial_\nu \Gamma^\alpha{}_{\mu\beta} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\gamma{}_{\nu\beta} - \Gamma^\alpha{}_{\nu\gamma}\Gamma^\gamma{}_{\mu\beta}}$$

(4.3) Notice that $\boxed{R^\alpha{}_{\beta\mu\nu} = -R^\alpha{}_{\beta\nu\mu}}$

which encodes the fact that traversing the loop in the above figure in the opposite direction gives the 'opposite' answer i.e. with a minus sign (area element is oriented oppositely.)

On a general tensor (for torsion-free connection) we can derive

(5.1)
$$\boxed{\begin{aligned}[\nabla_\alpha, \nabla_\beta] X^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = {} & R^{\mu_1}{}_{\gamma\alpha\beta} X^{\gamma\mu_2\cdots\mu_k}{}_{\nu_1\cdots\nu_l} \\ & + R^{\mu_2}{}_{\gamma\alpha\beta} X^{\mu_1\gamma\mu_3\cdots\mu_k}{}_{\nu_1\cdots\nu_l} + \cdots \\ & - R^{\gamma}{}_{\nu_1\alpha\beta} X^{\mu_1\cdots\mu_k}{}_{\gamma\nu_2\cdots\nu_l} \\ & - R^{\gamma}{}_{\nu_2\alpha\beta} X^{\mu_1\cdots\mu_k}{}_{\nu_1\gamma\nu_3\cdots\nu_l} - \cdots\end{aligned}}$$

Other properties of Riemann:-

(5.2)
$$\boxed{R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}}$$
$$\boxed{R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}}$$
$$\boxed{R_{\rho[\sigma\mu\nu]} = 0}$$
$$\boxed{R_{[\rho\sigma\mu\nu]} = 0}$$
$$\boxed{\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0}$$

Ricci tensor, Ricci scalar, Einstein tensor

(5.3) $$\boxed{R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu}}$$

(5.4) $$\boxed{R \equiv R^{\mu}{}_{\mu}}$$

(5.5) $$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}$$

Then

(5.6) $$\boxed{\nabla^{\mu} G_{\mu\nu} = 0}$$

Killing Vectors

Along a geodesic, we have that $p^\lambda \nabla_\lambda p^\mu = 0$, a tensor equation which can therefore be rewritten

$$p^\lambda \nabla_\lambda p_\mu = 0 = p^\lambda \partial_\lambda p_\mu - p^\lambda \Gamma^\nu{}_{\lambda\mu} p_\nu$$

so
$$\begin{aligned} p^\lambda \partial_\lambda p_\mu &= \Gamma^\nu{}_{\lambda\mu} p^\lambda p_\nu \\ &= \tfrac{1}{2} g^{\nu\sigma} (g_{\lambda\sigma,\mu} + g_{\mu\sigma,\lambda} - g_{\lambda\mu,\sigma}) p^\lambda p_\nu \\ &= \tfrac{1}{2} (g_{\lambda\sigma,\mu} + g_{\mu\sigma,\lambda} - g_{\lambda\mu,\sigma}) p^\lambda p^\sigma \\ &= \tfrac{1}{2} g_{\lambda\sigma,\mu} p^\lambda p^\sigma \end{aligned}$$

If we have a massive particle, we can choose proper time, so then $p^\lambda \frac{\partial}{\partial x^\lambda} = \left(m \frac{dx^\lambda}{d\tau}\right) \partial_\lambda = m \frac{d}{d\tau}$

(6.1) i.e. $\boxed{g_{\lambda\sigma,\mu} = 0 \Rightarrow \frac{dp_\mu}{d\tau} = 0}$ Symmetry $\Rightarrow$ Conservation law

In general, if the vector K points in the direction for which (8.1) is true, it is possible to rewrite it in covariant terms as

(6.2) $\boxed{\nabla_{(\mu} K_{\nu)} = 0 \Rightarrow p^\mu \nabla_\mu (K_\nu p^\nu) = 0}$ K= Killing vector

There are also Killing tensors which satisfy
$$\nabla_{(\mu} K_{\nu_1 \dots \nu_\ell)} = 0 \Rightarrow p^\mu \nabla_\mu (K_{\nu_1 \dots \nu_\ell} p^{\nu_1} \dots p^{\nu_\ell}) = 0$$

Other useful identities involving Killing vectors are
$$\nabla_\mu \nabla_\sigma K^\alpha = R^\alpha{}_{\sigma\mu\beta} K^\beta \quad \leftarrow \text{(Symmetry + Killing eqn)}$$
and so
$$\nabla_\mu \nabla_\sigma K^\mu = R_{\sigma\beta} K^\beta \quad \leftarrow \text{(contracting)}$$
then $\boxed{K^\lambda \nabla_\lambda R = 0}$ $\leftarrow$ (Using Bianchi identity as well as contracting further)