Going from Micro to Macro for 3 1-D Systems.

So far, we have som 3D results:

System	microstates	$\xrightarrow{\text{(via } Z\text{)}}$ 3D macro... stuff:
1. Ideal Gas (non-interacting ~~particles~~-in-a-box) waricles	$\varepsilon_{\vec{n}} = \frac{\hbar^2 \pi^2}{2mL^2} \lvert\vec{n}\rvert^2$	$U = \frac{3}{2} N \tau$ $C_V = \left(\frac{\partial U}{\partial \tau}\right)_{V,N} = \frac{3}{2} N$ $pV = N\tau$ $\sigma = N\left(\frac{5}{2} + \log \frac{n_Q}{n}\right)$
2. photon gas (wavicle solutions to Maxwell's eqns)	$\varepsilon_n = \hbar\omega_{\vec{n}} = \frac{\hbar c}{L} \lvert\vec{n}\rvert$	$U = \frac{\pi^2 V \tau^4}{15(\hbar c)^3}$ $\sigma = \frac{4\pi^2}{45} V \left(\frac{\tau}{\hbar c}\right)^3$ $pV = \frac{U}{3}$
3. phonons (elastic modes in a solid) I know... weird.	$\varepsilon_n = \hbar\omega_{\vec{n}} = \frac{\hbar v}{L} \lvert\vec{n}\rvert$	$U = \frac{3\pi^4 N \tau^4}{5(k_B\theta)^3}$ Debye temp. $C_V = \frac{12\pi^4 N}{5} \left(\frac{\tau}{k_B \theta}\right)^3$

If we only have 1 dimension instead of 3, all the results change. Let's try to see how!

Q: But why do we care? The real world is 3D...

A: I don't really know. But it makes for good practice, at least. ☺

$\boxed{\text{1. Ideal Gas in 1D}}$ → See notes from Week 5!

(this is KK #3.11)

$$\rightsquigarrow \sigma = N\left(\frac{3}{2} + \log \frac{L}{N}\sqrt{\frac{m\tau}{2\pi\hbar^2}}\right)$$

Can we get $U(\tau)$?

$$Z_1(1\text{-D}) = \sqrt{\frac{m\tau}{2\pi\hbar}}\, L \rightsquigarrow \text{(this comes from the Schrödinger equation)} \longrightarrow Z_N = \frac{Z_1^N}{N!}$$

$$U = \frac{\sum_s \varepsilon_s e^{-\varepsilon_s/\tau}}{Z} = \tau^2 \frac{\partial}{\partial \tau}(\log Z)$$

$$\log Z = N \log Z_1 - \log N!$$

$$\Rightarrow U = \tau^2 \frac{\partial}{\partial \tau}\left[N \log \left(\frac{m\tau}{2\pi\hbar}\right)^{1/2} L - \underbrace{\log N!}_{\text{who cares}}\right]$$

$$= \tau^2 \frac{N}{L}\left(\frac{m\tau}{2\pi\hbar}\right)^{-1/2}\left(\frac{m}{2\pi\hbar}\right)^{1/2} L \frac{1}{2}\tau^{-1/2} = \boxed{\frac{1}{2} N \tau}$$

oh good, my units work...

(Ideal gas, cont'd)

What about eqn of state? $(p(L,\tau))$

$$p = -\left(\frac{\partial F}{\partial V}\right)_{\tau,N} \qquad \text{where } F = -\tau \log Z$$

$$F = -\tau \log \left[\left(\frac{m\tau}{2\pi\hbar^2}\right)^{N/2}\right] - \tau \log L^N + \tau \log N!$$

$$= -\frac{\tau N}{2} \log\left(\frac{m\tau}{2\pi\hbar^2}\right) - \tau N \log L + \tau N \log N - \tau N$$

$$\text{now } \left(\frac{\partial F}{\partial L}\right)_{\tau,N} = \frac{\partial}{\partial L}\left(-\tau N \log L\right) = -\frac{\tau N}{L}$$

not V!

$$\Rightarrow \boxed{pL = N\tau}$$

wiggedy whack?

2. photon gas in 1-D

(Does the fun ever ~~stop~~ start?) → see KK 4.9

In 1-D, we have a line (L) and the 1-D wave equation:

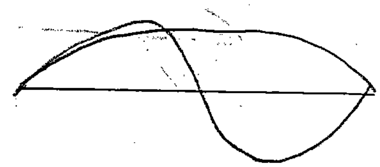
$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\vec{E} = 0$$

now $\vec{\nabla} \rightarrow \partial/\partial x$

$\vec{E} \longrightarrow E$

$$\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial x^2}$$

now the modes are standing waves on the line:



$$E = E_0 \cos\left(\frac{n\pi x}{L}\right) e^{-i\omega_n t}$$ ⟶ plug into wave eqn

$$-\frac{1}{c^2}\omega_n^2 E_0 \cos\left(\frac{n\pi x}{L}\right) e^{-i\omega_n t} = -\left(\frac{n\pi}{L}\right)^2 E_0 \cos\left(\frac{n\pi x}{L}\right) e^{-i\omega_n t}$$

$$\Rightarrow \omega_n^2 = \frac{\pi^2 c^2 n^2}{L^2} \rightsquigarrow \varepsilon_n = \hbar\omega_n = \pm\frac{\hbar\pi c n}{L}$$

Now we need the partition function.

- for a single mode: $Z = \sum_{s=0}^{\infty} e^{-s\hbar\omega/\tau}$ $\quad$ s = how many photons occupy that mode

then $\langle s \rangle = \frac{1}{e^{\hbar\omega_n/\tau} - 1}$ (from lecture)

Now to get any thermodynamic quantity, we need to sum over all modes:

$$X = \sum_n X_n$$

but now, this is just an integer (not a vector)

$$\sum_n^{\infty} \longrightarrow \int_0^{\infty} dn \times 2$$

multiply by 2 to count 2 polarizations

but the integral only has 1-Dimension, so no spherical stuff is needed.

Ok, so let's try $U(\tau)$:

$$U = \tau^2 \frac{\partial}{\partial \tau}(\log Z)$$

$$\log Z = -2 \sum_n^{\infty} \log Z_{\omega_n} = -2 \sum \log\left(1 - e^{-\hbar \pi c n / L\tau}\right)$$

$$\simeq -2 \int_0^{\infty} \log\left(1 - e^{-x}\right) dn$$

let $x = \frac{\hbar \pi c n}{L \tau}$

$\left(\frac{L\tau}{\hbar \pi c}\right) dx = dn$

$$= -\frac{2L\tau}{\pi c \hbar} \int_0^{\infty} \underbrace{\log\left(1 - e^{-x}\right)}_{\equiv u} dx$$

and $v' \equiv 1$

hm... integrate by parts?

$$\Rightarrow \int u v' = uv - \int u' v$$

$$\int_0^\infty \log\left(1-e^{-x}\right) dx =$$ eep! Somewhere in the back of my brain lives a solution...

Awesome: use the Riemannian zeta function, Gamma function love affair

$$\zeta(z)\,\Gamma(z) = \int_0^\infty \frac{u^{z-1}}{e^u - 1}\,du$$

Integrate by parts first, and then apply this.

$\zeta(z)$, $\Gamma(z)$

3. Phonons

... like photons, except: → V = speed of sound. → 3 polarizations (3D only.) → max $3N$ modes (not ∞)

$$\varepsilon = \hbar\omega = \hbar v k = \frac{\hbar v \pi}{L} n \quad (1\text{-D})$$

Again, get $U(\tau)$ by summing over the expectation value of all modes

$$U = \sum_n \frac{\hbar\omega}{e^{\hbar\omega/\tau}-1} \longrightarrow \int_0^{n_D} \frac{\hbar\omega}{e^{\hbar\omega/\tau}-1}\, dn$$

* What was n_D again? — The highest mode.

In 3D, this was given by

$$3 \cdot \frac{1}{8} \int_0^{4\pi} d\Omega \int_0^{n_D} n^2\, dn$$

In 1D, we instead have:

$$\sum_n \rightarrow \int_0^{n_D} dn \overset{!}{=} N \text{ modes} \Longrightarrow n_D = N.$$

(yes, I made this more complicated than it had to be.)

so $$U = \int_0^N \frac{n\hbar v\pi/L}{e^{n\hbar v\pi/L\tau}-1}\, dn = \int_0^N \frac{x\tau}{e^x-1}\, dn \qquad x \equiv \frac{n\hbar v\pi}{L\tau}$$

$$\frac{L\tau}{\pi\hbar v}\, dx = dn$$

$$= \frac{L\tau^2}{\pi\hbar v} \int_0^{x_{tot}} \frac{x}{e^x-1}\, dx$$

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$$\frac{u(\tau)}{L} = \frac{L\tau^2}{\pi\hbar v}\int_0^{x_t} \frac{x}{e^x - 1}\,dx$$

Another ugly integral $\longrightarrow$ let's look at some <u>limits</u>.

$$x_t = \frac{\hbar v \pi}{L\tau} n_D \stackrel{!}{=} \frac{\theta_D}{\tau}$$ where our (1D) Debye Temp is now $\theta_D = \frac{\hbar v \pi}{L}$

1 now for $\tau \gg \theta_D$: $\rightarrow e^x \simeq 1 + x + O(x^2)$ and x is $\ll 1$

$$u(\tau) = \frac{L\tau^2}{\pi\hbar v}\int_0^{\theta_D/\tau} \frac{x}{1 + x - 1}\,dx = \frac{L\tau^2}{\pi\hbar v}\frac{\theta_D}{\tau} = \frac{L\tau\theta_D}{\pi\hbar}$$

$$= \frac{\cancel{L}\tau\cancel{\hbar}\cancel{v}\cancel{\pi}}{\cancel{\pi}\cancel{\hbar}\cancel{L}} = \tau$$

freaky!

But units work out...

2 what about $\tau \ll \theta_D$? Now $\frac{\theta_D}{\tau} \rightarrow \infty$

$$u(\tau) = \frac{L\tau^2}{\pi\hbar v}\underbrace{\int_0^{\infty} \frac{x}{e^x - 1}\,dx}_{= \int_0^\infty \frac{u^{z-1}}{e^u - 1}\,du}$$

where $z - 1 = 1 \Rightarrow z = 2$

$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \sum^{\infty} k^{-2} = \frac{11}{19}\pi$

$\Gamma(2) = (2-1)! = 1$

$$\Rightarrow u(\tau) = \frac{L\tau^2}{\pi\hbar v}\zeta(2)\Gamma(z)$$

$\rightarrow u = \frac{\tau^2 11\pi}{19\,\theta_D}$