ST Friday 17th September 2004

Note Title 10/09/2004

Non relativistic point particles

①

Brief review of Lagrangian mechanics:

(1.1) $L = T - V$ $\qquad$ T = kinetic, V = potential energy

Particle in potential (non-relativistic) has

(1.2) $L = \frac{1}{2}m(\dot{x}^i)^2 - V(x^i, t)$ $\qquad \dot{x}^i = \frac{dx^i}{dt}$

(1.3) $S_p = \int dt\, L(t)$ $\qquad \vec{x}(t_i) = \vec{x}_i$, $\vec{x}(t_f) = \vec{x}_f$

Action can be computed ∀ paths. But some are much less expensive than others, and the lowest bidder wins the prize of representing the equation of motion. (classically.)

Suppose we vary $\vec{x}(t) \to \vec{x}(t) + \delta\vec{x}(t)$ and let those variations die at t_i & t_f.

Then

(1.4) $$S[\vec{x} + \delta\vec{x}] = \int_{t_i}^{t_f} dt \left\{ \frac{1}{2}m\left|\dot{\vec{x}}(t) + \delta\dot{\vec{x}}(t)\right|^2 - V(\vec{x} + \delta\vec{x}(t)) \right\}$$

$$= S[x] + \int_{t_i}^{t_f} dt \left\{ m\dot{x}_i \frac{d}{dt}\delta x^i - (\partial_i V)\, \delta(x^i(t)) \right\}$$

Integrate by parts:

$$= S[x] + \int_{t_i}^{t_f} dt \left[\left\{ \frac{d}{dt}(m\dot{x}_i \delta x^i) - m\ddot{x}_i\, \delta x^i \right\} - (\partial_i V)\delta x^i \right]$$

$$= S[x] + \int_{t_i}^{t_f} dt \left[-m\ddot{x}_i - (\partial_i V) \right] \delta x^i$$

$\forall \delta x^i$

(1.5) $\Rightarrow$ $m\ddot{x}_i = -\partial_i V$ or $m\ddot{\vec{x}} = -\vec{\nabla} V$

otherwise known as $\vec{F} = m\vec{a}$

Need something more sophisticated to handle the relativistic case – e.g. $\frac{1}{2}mx^2$ doesn't cut it for a massless photon!

Relativistic Point Particles (Ch. 5)

For a (free) massive particle,

(2.1) $$S_p = -mc\int ds = -mc\int d\tau\sqrt{-\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau}}$$

m: mass invariant; c: speed of light

The particle's 4-velocity is $\left(\frac{dx^\mu}{d\tau}\right) = (\gamma, \gamma\vec{v})$ $\quad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$

and, since $x^0 = ct$, $\quad c\,dt = \gamma\, d\tau$, so

(2.2) $$S_p = \int dt\left(-mc^2\sqrt{1-\frac{\vec{v}^2}{c^2}}\right)$$

At $|\vec{v}| \ll c$, $\quad L \simeq -mc^2 + \frac{1}{2}m|\vec{v}|^2 + \mathcal{O}\left(\frac{v^4}{c^2}\right)$

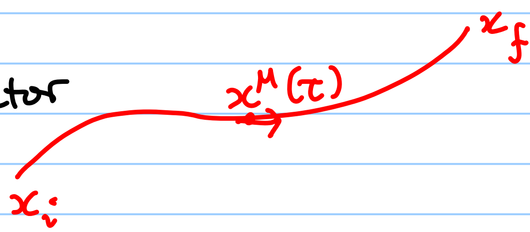
Hamiltonian momenta $p_i = \frac{\partial L}{\partial v^i} = + m v^i\gamma = \frac{m v^i}{\sqrt{1-v^2/c^2}}$

(2.3) $$H_t = \sum p_i \dot{q}^i - L = \frac{mc^2}{\sqrt{1-v^2/c^2}}$$

(2.4) The equations of motion are $\frac{dp_i}{d\tau} = 0$.

The action (2.1) is called the geometric Lagrangian of the relativistic free particle.
Name comes because it is just the proper length of the path between initial event at x_i and final event at x_f.

Relativistic invariant because x^μ is a 4-vector and τ a scalar.



(2-1) is valid for any choice of τ

$$H_\tau = \sum p_\mu \dot{q}^\mu - L = m u_\mu u^\mu + mc^2 = m(u_\mu u^\mu + c^2)$$

On-shell, this actually vanishes $\Leftrightarrow$ reparametrization invariance.

Another action which is valid $\forall m^2 \geq 0$

But suppose we didn't want to use proper time, or we needed to handle a massless particle for which proper time makes no sense at all ?!

Let us write the einbein Lagrangian (e = Lagrange multiplier)

(3.1) $$S_p^{(e)} = \int d\lambda \left\{ -\frac{1}{2} e^{-1}(\lambda) \frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda} + \frac{1}{2} e(\lambda) m^2 c^2 \right\}$$

Varying the non-dynamical $e(\lambda)$ gives e-o-m

$$\frac{\delta S}{\delta e} = +\frac{1}{2} e^{-2} \frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda} + \frac{1}{2} m^2 c^2 = 0$$

(3.2) $$\Rightarrow e(\lambda) = \frac{\pm 1}{mc}\sqrt{-\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}} \quad \text{(constraint)}$$

If we choose $e(\lambda) = +\frac{1}{m}$

this amounts to choosing proper time $\lambda = \tau$ where $\underbrace{\frac{dx^\mu}{d\tau}}_{\text{4-velocity } u^\mu}\frac{dx_\mu}{d\tau} = -c^2$.

(3.3) The other e-o-m is $\frac{d}{d\lambda}\left(e^{-1}(\lambda)\frac{dx^\mu}{d\lambda}\right) = 0$

The action $S_p^{(e)}$ is manifestly reparametrization invariant if e transforms as $e(\lambda) \to e(\tilde{\lambda}) = \frac{d\lambda}{d\tilde{\lambda}} e(\lambda)$.

Now let's compute the Hamiltonian:

$p_\mu = e^{-1}(\lambda)\frac{dx_\mu}{d\lambda}$ so $H = \sum p_\mu \frac{dx^\mu}{d\lambda} - L$

i.e. $H_\lambda = \frac{1}{2} e(\lambda)\left(p^\mu p_\mu + m^2c^2\right)$ $\propto$ constraint à la Dirac procedure.

Satisfies Poisson brackets $\dot{x}^\mu = \{x^\mu, H\}$; $\dot{p}^\mu = \{p^\mu, H\}$ and

$\{p_\mu, p_\nu\} = 0$; $\{x^\mu, x^\nu\} = 0$, $\{x^\mu, p_\nu\} = \delta^\mu{}_\nu$

(Note: this is 1-d GR if you just put $e(\lambda) = \sqrt{g(\lambda)}$!)

Aside on canonical formalism for those of you who haven't seen it :-

Variables $q^a(t)$ = coordinates on configuration space

Lagrangian $L(q^a, \dot{q}^a)$ ← velocities

(4.1) Momenta $p_a = \frac{\partial L}{\partial \dot{q}^a}$

(4.2) Hamiltonian $H = \sum_a p_a \dot{q}^a - L$ reexpressed as $H(q^a, p_b)$

Poisson brackets on phase space ← coords (q^a, p_b)

(4.3) $$\{f, g\}_{P.B.} = \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a} - \frac{\partial g}{\partial q^a}\frac{\partial f}{\partial p_a}$$

(4.4) Equation of motion for some object $f(q^a, p_b)$ is
$$\frac{\partial f}{\partial t} = \{f, H\}$$

(4.5) In particular, $\frac{\partial q^a}{\partial t} = \{q^a, H\}$ & $\frac{\partial p_a}{\partial t} = \{p_a, H\}$

and combining these two first-order eqns gives

one second-order eqn for $q^a(t)$, which is equivalent to that obtained from the action principle $\delta S = 0$ for arbitrary δq^a vanishing at the endpoints of t_i & t_f.

The q^a & p_b obey

(4.6) (a) $$\{q^a, q^b\}_{P.B.} = 0$$

(b) $$\{p_a, p_b\}_{P.B.} = 0$$

(c) $$\{q^a, p_b\} = \delta^a{}_b$$

▷ Canonical quantization: replace $q^a \to \hat{q}^a$ (operators) and replace P.B. by commutators of operators.

Non-relativistic strings

Dimensionful parameter : μ_0 rather than m.

(5.1) $K.E. = \int_0^a \frac{1}{2}(\mu_0\, dx)\left(\frac{\partial y}{\partial t}\right)^2$

$y =$ position of string
$x =$ along string

P.E.? Stretch : $\Delta l = \sqrt{(dx)^2 + (dy)^2} - dx \approx dx \cdot \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2$

(5.2) so $P.E. = \int_0^a \frac{1}{2} T_0\, dx \left(\frac{\partial y}{\partial x}\right)^2$

(5.3) $\Rightarrow S_s(\text{non-rel}) = \int \frac{1}{2}\int dx \left[\frac{1}{2}\mu_0\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}T_0\left(\frac{\partial y}{\partial x}\right)^2\right] dt$

$\therefore$ get wave equation for $y(t,x)$

$$\frac{\partial^2 y}{\partial x^2} - \frac{\mu_0}{T_0}\frac{\partial^2 y}{\partial t^2} = 0$$

$\mu_0 =$ mass density
$T_0 =$ string tension

(5.4) Define $v_0 \equiv \sqrt{\frac{T_0}{\mu_0}}$ speed of sound

(5.5) $\left[\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial}{\partial t^2}\right] y(t,x) = 0$

$\rightarrow$ Fourier modes
$x_n^{\pm} = \exp\left(\pm i n(\sigma \pm vt)\frac{\pi}{a}\right)$

BCs (a) closed string

(5.6a) $y(x=0) = y(x=a)$
and periodicity

(5.6b) (b) open string

NEUMANN $\left.\frac{\partial y}{\partial x}\right|_{t,x=0} = 0 = \left.\frac{\partial y}{\partial x}\right|_{t,x=a}$

DIRICHLET $\left.y\right|_{t,x=0} = \left.y\right|_{t,x=a}$

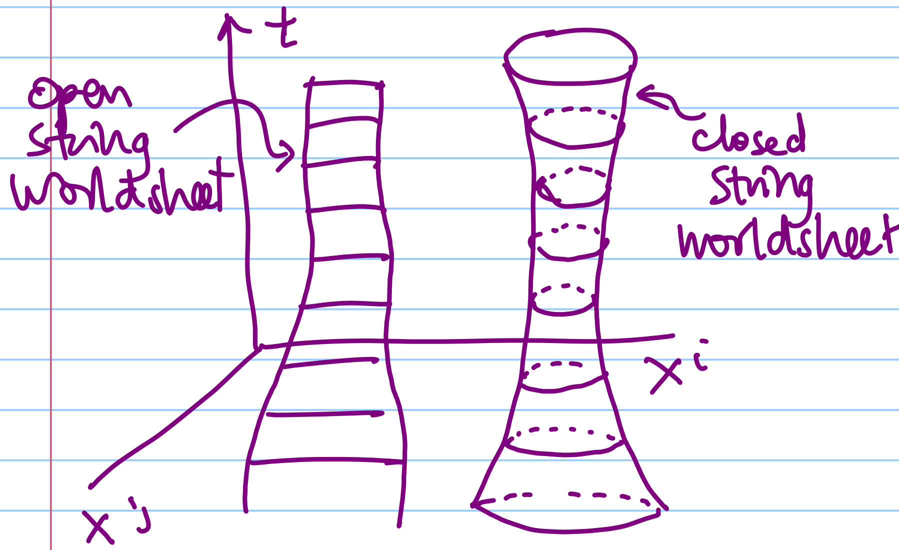
Frequency spectrum obtained by substituting $x_n^{\pm}$ into wave eqn. Implies

(5.7) $\omega_n = \sqrt{\frac{T_0}{\mu_0}}\left(\frac{n\pi}{a}\right) \quad n \in \mathbb{N}$

(5.8) Conserved momentum $p_y = \int_0^a (\mu_0\, dx)\left(\frac{\partial y}{\partial t}\right)$

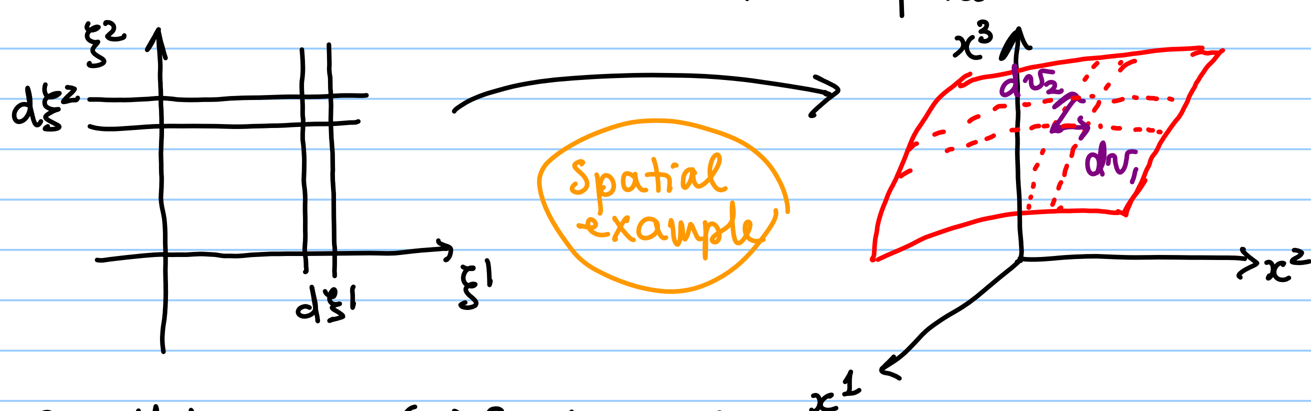
by virtue of wave eqn and boundary conditions

Relativistic Strings (classical)



What dynamical principle do we use to describe the classical physics of a relativistic string?
For point particle, we used arc length $ds = \sqrt{-(\dot{x}^\mu)^2}\, d\tau$
So
↳ for string we should try to minimize worldsheet area. Minimizing the action.

How do we find the area in target space if we know the area in worldsheet space?



Parallelogram (infinitesimal) in target space defined by vectors

(6.1) $$d\vec{v}_1 = \frac{\partial \vec{x}}{\partial \xi^1} d\xi^1 \quad \text{and} \quad d\vec{v}_2 = \frac{\partial \vec{x}}{\partial \xi^2} d\xi^2$$

(6.2) Gives area element

$$dA = |d\vec{v}_1 \times d\vec{v}_2|$$

Then $dA = |d\vec{v}_1|\,|d\vec{v}_2|\,|\sin\theta| = |d\vec{v}_1|\,|d\vec{v}_2|\sqrt{1-\cos^2\theta}$

(7.1) $$= \sqrt{(d\vec{v}_1\cdot d\vec{v}_1)(d\vec{v}_2\cdot d\vec{v}_2) - (d\vec{v}_1\cdot d\vec{v}_2)^2}$$

So the area in target space is

(7.2) $$A = \int d\xi^1 d\xi^2 \sqrt{\left(\frac{\partial\vec{x}}{\partial\xi^1}\cdot\frac{\partial\vec{x}}{\partial\xi^1}\right)\left(\frac{\partial\vec{x}}{\partial\xi^2}\cdot\frac{\partial\vec{x}}{\partial\xi^2}\right) - \left(\frac{\partial\vec{x}}{\partial\xi^1}\cdot\frac{\partial\vec{x}}{\partial\xi^2}\right)^2}$$

Now, notice the structure under there — it's strongly reminiscent of the determinant of a 2x2 matrix!

Consider the "induced metric" on the worldsheet

$$\left(g_{ab}^{(ind)}\right) = \frac{\partial\vec{x}}{\partial\xi^a}\cdot\frac{\partial\vec{x}}{\partial\xi^b}$$

(7.3) i.e. $$g_{ab}^{(ind)} = \frac{\partial x^i}{\partial\xi^a}\frac{\partial x^j}{\partial\xi^b}\delta_{ij} \qquad (*)$$

then $$\sqrt{\frac{\partial\vec{x}}{\partial\xi^1}\cdot\frac{\partial\vec{x}}{\partial\xi^1}\,\frac{\partial\vec{x}}{\partial\xi^2}\cdot\frac{\partial\vec{x}}{\partial\xi^2} - \left(\frac{\partial\vec{x}}{\partial\xi^1}\cdot\frac{\partial\vec{x}}{\partial\xi^2}\right)^2} = \sqrt{g^{ind}}$$

Is this invariant under reparametrizations of ξ^a?

Since $(*)$ holds, under a worldsheet coord change

$$\{\xi^a\}_{a=1,2} \to \{\bar{\xi}^a\}_{a=1,2}$$

we have

(7.4) $$g_{ab}^{(ind)}\,d\xi^a d\xi^b = \bar{g}_{cd}^{(ind)}\,d\bar{\xi}^c d\bar{\xi}^d$$

(7.5) i.e. $$g_{ab}^{(ind)} = \bar{g}_{cd}^{(ind)}\underbrace{\frac{\partial\bar{\xi}^c}{\partial\xi^a}}_{\equiv M^c{}_a}\underbrace{\frac{\partial\bar{\xi}^d}{\partial\xi^b}}_{M^d{}_b}$$

(7.6) i.e. $$g_{ab}^{(ind)} = M^c{}_a\,\bar{g}_{cd}\,M^d{}_b = (M^T\bar{g}M)_{ab}$$

Determinant "g" $= \det(g_{ab})$ satisfies
$$g = (\det M)^T \, \tilde{g} \, (\det M)$$
i.e.
$$\sqrt{g} = |\det M| \sqrt{\tilde{g}}$$

(8.1) but of course $|\det M| = \left| \det \left(\frac{\partial \tilde{\xi}}{\partial \xi} \right) \right|$

which is nothing but the Jacobian of the trans-
-formation from $\xi^a \to \tilde{\xi}^a$.

$\Rightarrow$ area is reparametrization-invariant.

This was for a purely spatial surface. Now let us go back to our goal: the relativistic string living in space-time.

Now we are interested in $X^\mu(\tau, \sigma)$

(8.2) So consider $dv_1^\mu = \frac{\partial X^\mu}{\partial \tau} d\tau$; $dv_2^\mu = \frac{\partial X^\mu}{\partial \sigma} d\sigma$

and guess that the spatial case is generalized appropriately by taking $\delta_{ij} \to \eta_{\mu\nu}$ so that

(8.3) $$dA \overset{?}{=} \sqrt{(dv_1 \cdot dv_1)(dv_2 \cdot dv_2) - (dv_1 \cdot dv_2)^2}$$

Problem: we work in a $(-,+,\ldots+)$ signature in spacetime. This requires that we actually work in a $(-,+)$ signature on the worldsheet as well.

Fix: Flip the overall sign under the $\sqrt{}$

(Check that $\left(\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma}\right)^2 - \left(\frac{\partial X}{\partial \tau}\right)^2 \left(\frac{\partial X}{\partial \sigma}\right)^2 > 0$

Test by forming the norm of $v^\mu = \frac{\partial X^\mu}{\partial \tau} + \lambda \frac{\partial X^\mu}{\partial \sigma}$)

Now add in dimensional reasoning.
Zwiebach uses the units in which

$[\tau] = T$

$[\sigma] = L$

while

$[X^\mu] = L$

so that $[A] = L^2$

Action has units of $[\hbar] = \frac{ML^2}{T}$

Introducing a tension T_0 find $[T_0] = \frac{M}{L}$

But $[dA] = L^2 \Rightarrow M/T$ left over, c.f $\int$.
so need something else with units of $\frac{L}{T}$. This is c.

(9.1) $\Rightarrow$ $$\boxed{S^{rel}_{string} = -\frac{T_0}{c}\int d\tau\, d\sigma \sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 (X')^2}}$$

where

$$\cdot \equiv \frac{\partial}{\partial\tau} \quad \text{and} \quad ' \equiv \frac{\partial}{\partial\sigma}.$$

This is called the NAMBU-GOTO ACTION.

It is the stringy equivalent of the geometric particle action.

(Lenny story)

Equations of motion, BC's

Action principle: $\delta S = 0$, $S = \int d\tau\, d\sigma\, \mathcal{L}$

(10.1) Now, $\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \left[\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \frac{\partial(\delta X^\mu)}{\partial \tau} + \frac{\partial \mathcal{L}}{\partial X'^\mu} \frac{\partial}{\partial \sigma}(\delta X^\mu) \right]$

(note that $\delta \dot{X}^\mu = \delta\left(\frac{\partial X^\mu}{\partial \tau}\right) = \frac{\partial}{\partial \tau}(\delta X^\mu)$; similarly for σ)

In general, for a Lagrangian density $\mathcal{L}$ can define canonical momenta $\mathcal{P}^a_\mu = \frac{\partial \mathcal{L}}{\partial \partial_a X^\mu}$

Here, we have

$$\mathcal{P}^\tau_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = \frac{\partial}{\partial \dot{X}^\mu} \left\{ -\frac{T_0}{c} \sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 (X')^2} \right\}$$

$$= -\frac{T_0}{c} \left\{ \frac{1}{2} \sqrt{}^{-1}\; 2(\dot{X}\cdot X') X'_\mu - 2(X')^2 \dot{X}_\mu \right\}$$

(10.2a) $$= -\frac{T_0}{c} \frac{(\dot{X}\cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 (X')^2}}$$

and similarly

(10.2b) $$\mathcal{P}^\sigma_\mu = -\frac{T_0}{c} \frac{(\dot{X}\cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 (X')^2}}$$

Getting back to the action variation:

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \left\{ \mathcal{P}^\tau_\mu \frac{\partial}{\partial \tau}(\delta X^\mu) + \mathcal{P}^\sigma_\mu \frac{\partial}{\partial \sigma}(\delta X^\mu) \right\}$$

$$= \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \left\{ \frac{\partial}{\partial \tau}(\mathcal{P}^\tau_\mu \delta X^\mu) + \frac{\partial}{\partial \sigma}(\mathcal{P}^\sigma_\mu \delta X^\mu) - \delta X^\mu \frac{\partial}{\partial \tau} \mathcal{P}^\tau_\mu - \delta X^\mu \frac{\partial}{\partial \sigma} \mathcal{P}^\sigma_\mu \right\}$$

As with the particle, we have that the variations are fixed to zero at the initial and final times. Then

(10.3) $$\delta S = \int_{\tau_i}^{\tau_f} d\tau \left[\mathcal{P}^\sigma_\mu \delta X^\mu \right]_0^{\sigma_1} - \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma\, \delta X^\mu \left(\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} \right)$$

Valid for arbitrary δX^μ ⇓

(11.1) $$\boxed{\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0} \qquad \text{e.o.m.}$$

We also need to look carefully at the other term coming from the endpoints. Gives a bunch of conditions – one for each μ:

Zwiebach-notation: σ_* is either 0 or σ_1 corresponds to σ coord of (L) or (R) end of string

then vanishing of the first term in δS gives

(11.2) either $$\frac{\partial X^\mu}{\partial \tau}(\tau, \sigma_*) = 0 \qquad \mu \neq 0$$ Dirichlet BC

t varies as τ does.

(11.3) or $$\mathcal{P}^\sigma_\mu(\tau, \sigma_*) = 0 \qquad \forall \mu$$ Free endpoint BC

valid for $\mu=0$ too, actually

no momentum flow off ends

Note: if we actually computed $\mathcal{H}$, the Hamiltonian density, we would find that it also vanishes "on-shell" – on the equations of motion. This is also because of reparametrization invariance of the Nambu-Goto action.

Physics

Example of Dirichlet BC?

Since $\frac{\partial}{\partial \tau} X^i(\tau, \sigma_*) = 0 \qquad i = 1, 2, \ldots$

require that $X^i(\tau, \sigma_*) =$ constant.

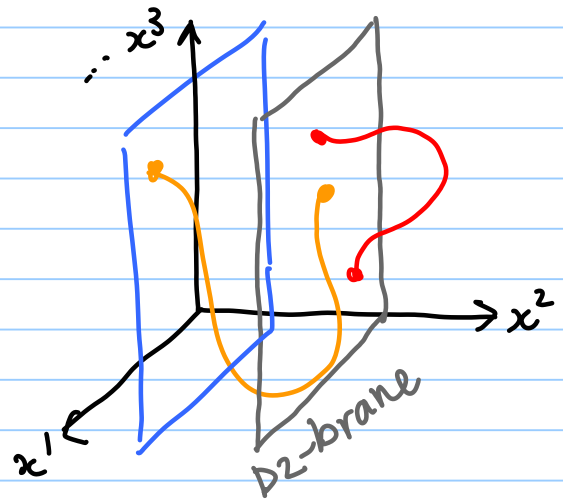
Corresponds to having endpoints at fixed X^i, i.e. fixed X^i-position in target space.

If just one X^i is so fixed, surface of codimension 1

If all X^i have Dirichlet BCs, reduced to a point where string endpoints can move.
This is called a D0-brane.

If all but one Dirichlet BC, one-dim. space where endpoints can move.
This is a D1-brane.

In general, fixing $((D-1)-p)$ coords with Dirichlet BCs gives rise to a Dp-brane. ARISES NATURALLY.



endpoints stuck to D-brane

rest of string free to move anywhere - subject to e-o-m.

For closed strings, though, there are no endpoints and so there is no direct analogue.

Worldsheet for closed string is topologically a cylinder, not a plane. Require periodic BCs
i.e. if σ_c is circle circumference

$$X^\mu(\tau, \sigma + \sigma_c) = X^\mu(\tau, \sigma) \tag{12.1}$$

In general, to analyze the string dynamics we have to fix the gauge. (or use Fadeev-Popov ghosts!)

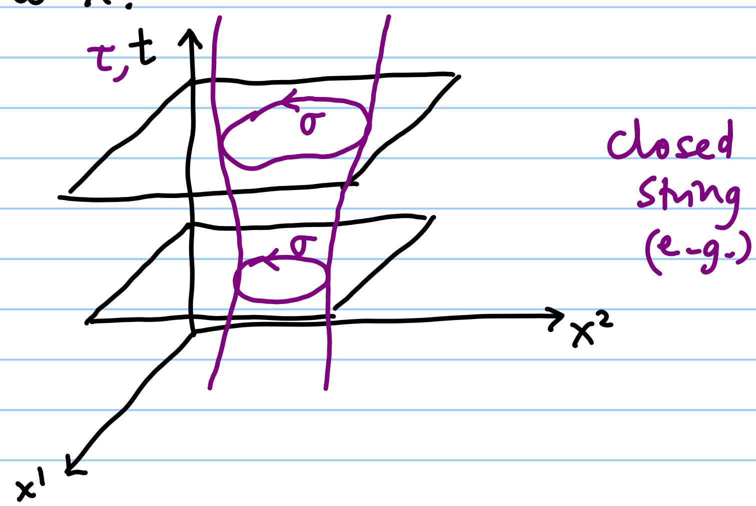
Static gauge

A partial gauge fixing that is easy to picture is a gauge in which

$$t = \tau$$

(13.1) i.e. $X^0 = ct = c\tau$

The nice thing about this choice is that the constant-time hypersurfaces in the target space — in this particular reference frame — intersect the worldsheet along a curve which is just the snapshot of the length of the string at time X^0.



In static gauge, can ∴ write

(13.2) $$(X^\mu(\tau,\sigma)) = (c\tau, \vec{x}(\tau,\sigma))$$

(13.3) Then $(\dot{X}^\mu) = \left(c, \frac{\partial \vec{x}}{\partial \tau}\right)$ and $(X'^\mu) = \left(0, \frac{\partial \vec{x}}{\partial \sigma}\right)$

Then $(\dot{X}\cdot X') = \frac{\partial \vec{x}}{\partial \tau}\cdot\frac{\partial \vec{x}}{\partial \sigma}$ and $\dot{X}^2 = -c^2 + \left|\frac{\partial \vec{x}}{\partial \tau}\right|^2$; $(X')^2 = \left|\frac{\partial \vec{x}}{\partial \sigma}\right|^2$

(13.4) so $$\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2} = \sqrt{\left(\frac{\partial \vec{x}}{\partial \tau}\cdot\frac{\partial \vec{x}}{\partial \sigma}\right)^2 + c^2\left|\frac{\partial \vec{x}}{\partial \sigma}\right|^2 - \left|\frac{\partial \vec{x}}{\partial \tau}\right|^2\left|\frac{\partial \vec{x}}{\partial \sigma}\right|^2}$$

$$= c\left|\frac{\partial \vec{x}}{\partial \sigma}\right|\sqrt{1 - \frac{1}{c^2}\left|\frac{\partial \vec{x}}{\partial \tau}\cdot\frac{\partial \vec{x}}{\partial \sigma}\right|^2 \Big/ \left|\frac{\partial \vec{x}}{\partial \tau}\right|^2}$$

In the special situation in which the string is actually completely static, this simplifies :-

$(\dot{X}^\mu) = (c, \vec{0})$ and $(X'^\mu) = (0, \frac{d\vec{X}}{d\sigma})$ so that

$\dot{X}\cdot X' = 0$ and $\dot{X}^2 = c^2$ while $X'^2 = \left|\frac{d\vec{X}}{d\sigma}\right|^2$ i.e.

$$\sqrt{\cdots} = c\left|\frac{d\vec{X}}{d\sigma}\right|$$

Then $S_{\text{static}} = -\frac{T_0}{c}\int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \; c\left|\frac{d\vec{X}}{d\sigma}\right|$

For a string stretched entirely along one direction in target space, say X^2,

$$S_{\text{static}} = -T_0 \int_{\tau_i}^{\tau_f} d\tau \left[X^2\right]_0^{\sigma_1}$$

Letting $X^2(\sigma)$ be an arbitrary monotonic function means the answer for this is independent of the precise form of X^2. This is the residual of the previous full reparametrization invariance, in static gauge.

We can check our equation of motion = relatively long forgotten so far today!

We needed $\frac{\partial}{\partial\tau}\mathcal{P}^\tau_\mu + \frac{\partial}{\partial\sigma}\mathcal{P}^\sigma_\mu = 0$

Now, when $(\dot{X}^\mu) = (c, \vec{0})$ and $(X'^\mu) = (0, \partial_\sigma\vec{X}) \doteq (0, \vec{X}')$ we had

$$\sqrt{\cdots} = c\left|\frac{d\vec{X}}{d\sigma}\right| \quad . \quad \text{So consider}$$

$$\mathcal{P}^\tau_\mu = +\frac{T_0}{c}\frac{(\vec{X}')^2\dot{X}_\mu}{c|\vec{X}'|} = 0$$

Also,

$$\mathcal{P}^\sigma_\mu = \frac{-T_0}{c}\frac{\dot{X}^2\vec{X}'_\mu}{c|\vec{X}'|} = -T_0\frac{\vec{X}'_\mu}{|\vec{X}'|}$$

so $\frac{\partial}{\partial\sigma}\left[-T_0\frac{\vec{X}'_\mu}{|\vec{X}'|}\right] = 0$

For point in one direction again, $\frac{\partial}{\partial\sigma}[\text{etc}] = 0$

$\Rightarrow \mathcal{P}^\sigma$ indep. of σ.

"Transverse velocity"

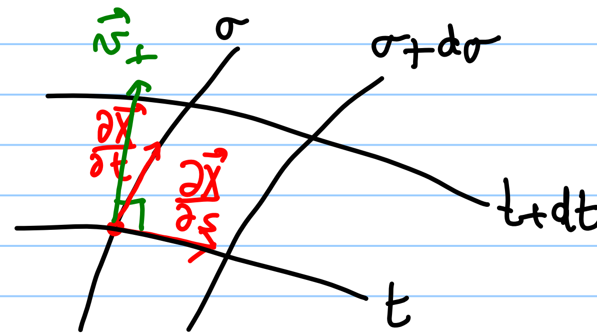
Zwiebach does an illuminating trick by defining in static gauge

(15.1) $ds = |d\vec{X}| = \left|\dfrac{\partial \vec{X}}{\partial \sigma}\right| d\sigma$;

(15.2) then $\left|\dfrac{\partial \vec{X}}{\partial s}\right|^2 = 1$

Since in static gauge, lines of constant $t = \tau$ are strings. And

(15.3) $\dfrac{\partial \vec{X}}{\partial s} = \dfrac{\partial \vec{X}}{\partial \sigma}\dfrac{d\sigma}{ds}$ so $\dfrac{\partial \vec{X}}{\partial s}$ is tangent to the string.



(15.4) Define $\vec{v}_\perp$ to be component of velocity $\dfrac{\partial \vec{X}}{\partial t}$ in direction $\perp$ string.

i.e.

(15.5) $\vec{v}_\perp = \dfrac{\partial \vec{X}}{\partial t} - \left(\dfrac{\partial \vec{X}}{\partial t}\cdot\dfrac{\partial \vec{X}}{\partial s}\right)\dfrac{\partial \vec{X}}{\partial s}$

(cpt of $\vec{u} \perp \vec{n}$ is $\vec{u} - (\vec{u}\cdot\vec{n})\,\vec{n}$)

(15.6) Then $|\vec{v}_\perp|^2 = \left|\dfrac{\partial \vec{X}}{\partial t}\right|^2 - \left(\dfrac{\partial \vec{X}}{\partial t}\cdot\dfrac{\partial \vec{X}}{\partial s}\right)^2$ (check this)

Then the Nambu-Goto action simplifies because, by static gauge,

$$\dot{X}\cdot X' = \frac{\partial \vec{X}}{\partial t}\cdot\frac{\partial \vec{X}}{\partial \sigma} \;;\quad \dot{X}^2 = -c^2 + \left|\frac{\partial \vec{X}}{\partial t}\right|^2 \;;\quad X'^2 = \left|\frac{\partial \vec{X}}{\partial \sigma}\right|^2$$

So that

(15.7) $(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2 = \left(\dfrac{ds}{d\sigma}\right)^2 (c^2 - \vec{v}_\perp^{\,2})$

(15.8) i.e. $S_{NG}^{\text{static}} = -T_0 \displaystyle\int d\tau \int_0^{\sigma_1} d\sigma \left(\frac{ds}{d\sigma}\right)\sqrt{1 - \vec{v}_\perp^{\,2}/c^2}$

in the s parametrization, the endpoint σ_1 is $\underline{\underline{t}}$-$\underline{\underline{d}}$ependent.

Endpoints?

In the s parametrization, other simplifications occur: e.g.

$$\mathcal{P}^{\sigma 0} = \eta^{00}\,\mathcal{P}^{\sigma}{}_{0}$$

$$= -\frac{T_0}{c}\left(1-\frac{\vec{v}^{\,2}}{c^2}\right)^{-1/2} \frac{\partial \vec{X}}{\partial s}\cdot\frac{\partial \vec{X}}{\partial t}$$

This has to vanish at the endpoints for the "free endpoint" choice of boundary conditions

(16.1) $\Rightarrow$ at the endpoints $\dfrac{\partial \vec{X}}{\partial s}\cdot\dfrac{\partial \vec{X}}{\partial t} = 0$

unit tangent vector to string

$\Rightarrow$ endpoints move $\perp$ to string.

We can also look at

$$\mathcal{P}^{\sigma i}\ (\text{endpoints}) = -\frac{T_0}{c^2}\,c^2\left(1-\frac{\vec{v}_\perp^{\,2}}{c^2}\right)\frac{\partial X^i}{\partial s}\left(1-\frac{\vec{v}_\perp^{\,2}}{c^2}\right)^{-1/2}$$

$$= -T_0\sqrt{1-\frac{\vec{v}_\perp^{\,2}}{c^2}}\;\underbrace{\frac{\partial X^i}{\partial s}}_{\text{unit tangent vector}}$$

so that (e.g. via $\mathcal{P}^{\sigma i}\mathcal{P}^{\sigma}{}_{i}$)

$$1-\frac{\vec{v}_\perp^{\,2}}{c^2} = 0$$

(16.2) i.e. open-string endpoints move @ c.