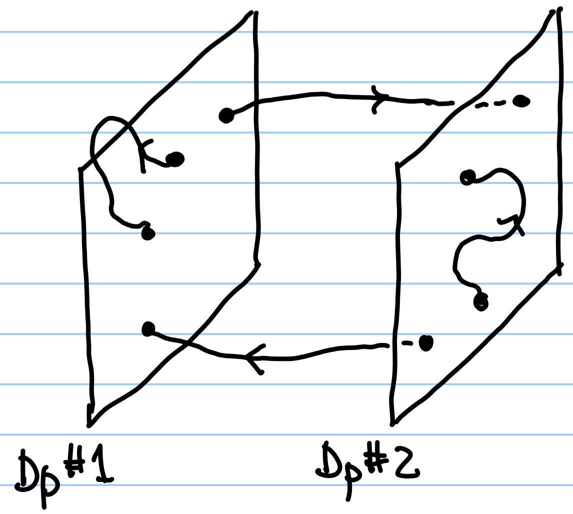
Open strings between parallel D-branes



4 types of open strings c.f. those we saw for a single Dp-brane

New ingredient = stretched strings between #1 & #2.

The NN coords still obey

(1.1) $$X^{m'}(\tau,\sigma)\big|_{\sigma=0} = 0 = X^{m'}(\tau,\sigma)\big|_{\sigma=\pi}, \quad m=0,1,\dots,p.$$

But now the DD coords satisfy

(1.2) $$X^a(\tau,\sigma)\big|_{\sigma=0} = \bar{x}_1^a, \quad X^a(\tau,\sigma)\big|_{\sigma=\pi} = \bar{x}_2^a, \quad a=p+1\dots d-1$$

In general, the solution is a superposition of L- and R-moving waves:

(1.3) $$X^a(\tau,\sigma) = \frac{1}{2}\left[f^a(\tau+\sigma) + g^a(\tau-\sigma)\right]$$

and applying the BC @ $\sigma=0$ gives

$$X^a(\tau,0) = \frac{1}{2}\left(f^a(\tau) + g^a(\tau)\right) = \frac{1}{2}\bar{x}_1^a$$

$$\Rightarrow \quad g^a(\tau) = -f^a(\tau) + 2\bar{x}_1^a, \quad \text{so that}$$

(1.4) $$X^a(\tau,\sigma) = \bar{x}_1^a + \frac{1}{2}\left[f^a(\tau+\sigma) - f^a(\tau-\sigma)\right]$$

Applying in turn the BC @ $\sigma=\pi$ gives

(1.5) $$2\bar{x}_1^a + f^a(\tau+\pi) - f^a(\tau-\pi) = 2\bar{x}_2^a$$

i.e.

(1.6) $$f^a(u+2\pi) - f^a(u) = 2(\bar{x}_2^a - \bar{x}_1^a)$$

$\Rightarrow f^{a\prime}$ periodic with period 2π

(2.1) so $f^a(u) = f_0^a u + \sum_{n=1}^{\infty}(h_n^a \cos(nu) + g_n^a \sin(nu))$

(any constant piece, $n=\emptyset$ mode, would just correspond to $2(\bar{x}_2^a - \bar{x}_1^a)$ ... so we leave it out)

Putting these equations together yields

(2.2) $$X^a(\tau,\sigma) = \bar{x}_1^a + (\bar{x}_2^a - \bar{x}_1^a)\frac{\sigma}{\pi} + \sum_{n=1}^{\infty}(f_n^a \cos n\tau + \tilde{f}_n^a \sin n\tau)\sin n\sigma$$

We can write this in terms of oscillator modes quantum mechanically:

(2.3) $$X^a(\tau,\sigma) = \bar{x}_1^a + (\bar{x}_2^a - \bar{x}_1^a)\frac{\sigma}{\pi} + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^a e^{-in\tau}\sin n\sigma$$

N.B. ⊛ these α_n^a are different to the oscillators used to quantize (1,1) (or (2,2)) strings; these ones are relevant to (1,2) (& (2,1)) guys!

(Note that $1 \leftrightarrow 2$ in the above is correct ☺)

So. These modes have the property that

(2.4) $$\dot{X}^a = -i\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^a e^{-in\tau}\sin n\sigma$$

(2.5) & $$X^{a\prime} = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^a e^{-in\tau}\cos n\sigma ,$$

where we have abbreviated

(2.6) $$\alpha_0^a \equiv \frac{1}{\sqrt{2\alpha'}}\frac{1}{\pi}(\bar{x}_2^a - \bar{x}_1^a) .$$

So...

(2.7) $$X^{a\prime} \pm \dot{X}^a = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^a e^{-in(\tau\pm\sigma)}$$

like we had before when quantizing the open string for the very first time.

$\Rightarrow$ CCR's for α's is same as before (but note ⊛!)

HW#2 problem 1 gets you to show explicitly that, regardless of N-or D-ness of X^i, X^- is still NN.

Using knowledge of $X^-(X^+ \propto \tau, X^i)$ from before, find that

(3.1) $$2p^+p^- = \frac{1}{\alpha'}\left[\alpha' p^i p^i + \frac{1}{2}\alpha_0^a \alpha_0^a + \sum_{n=1}^{\infty}(\alpha_{-n}^i \alpha_n^i + \alpha_{-n}^a \alpha_n^a) - 1\right]$$

$a = -1$ bosonic (e.g.)

where $I = (i, a)$

so that

$M^2 = 2p^+p^- - p^i p^i$ is given by

(3.2) $$\boxed{M^2 = \left(\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'}\right)^2 + \frac{1}{\alpha'}(N^\perp - 1)} \quad \text{(stretched strings)}$$

this has a beautiful physical interpretation: it is {the length of a straight (non-wiggly) string stretched between x_1^a and x_2^a} × {string tension}, plus what came before in terms of form of oscillator contributions:

(3.3) $$N^\perp = \sum_{n=1}^{\infty}\sum_{i=2}^{p} n (a_n^i)^\dagger (a_n^i) + \sum_{m=1}^{\infty}\sum_{a=p+1}^{d-1} m (a_m^a)^\dagger (a_m^a)$$

We then build up state space in much the same way as before. This is not convoluted for the (1,1) or (2,2) states. But a new wrinkle appears for the (1,2) & (2,1) string states: it can be confusing to wonder where these states actually live! The best answer is to say (like some Zen koan?) that they live on both D-branes: #1 and #2. In order to fully grok aspects of this story, one makes good use of a branch of math called non-commutative geometry. I will not have time to say much at all about NCG in this course.

Lowest-lying fields of stretched strings?

(3.4) For the bosonic string, (& no oscillators) $\quad M^2 = -\frac{1}{\alpha'} + \left(\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'}\right)^2 \geq -\frac{1}{\alpha'}$

This becomes zero at $|\bar{x}_2^a - \bar{x}_1^a| = 2\pi\sqrt{\alpha'}$.

At even larger separations, this scalar field becomes massive.

One oscillator

We can label the groundstate for the different kinds of strings by "[11], [22], [12], [21]" in Z. notation. Then have states

(4.1) $$a_1^{a\dagger} | p^+, \vec{p}_T ; [12] \rangle \quad , \quad a = p+1, \dots, d-1$$

Under $so(1,p)$ these transform as (Lorentz) scalars.

There are also the vectors under $so(1,p)$

(4.2) $$a_1^{i\dagger} | p^+, \vec{p}_T ; [12] \rangle \quad , \quad i = 2, \dots, p$$

Both sets of fields have mass given by

(4.3) $$M^2 = \left| \frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'} \right|^2 + 0 \quad \geq 0$$

- How many independent d.o.f. are there, here?

(4.4) $$\begin{cases} \perp \text{ to branes : } (d-1) - p = d-p-1 & \text{massive scalars} \\ \parallel \text{ to branes : } (p+1) - 2 = p-1 & \underbrace{\text{massive}}_{\substack{\text{for generic} \\ \text{separation} \\ \text{along } x^a \dots}} \text{ vectors} \end{cases}$$

In D spacetime dimensions, a massive vector has $(D-1)$ independent components for each value of p_μ as compared to $(D-2)$ for a massless vector; this corresponds to the longitudinal polarization.

c.f.
- sound in $D=4$: 3 modes
- light in $D=4$: 2 modes

So one of the scalars must join forces with the 'vectors' above to make a real live massive vector field, and leave the other $(d-p-2)$ scalars to actually be scalars.

Which one is it?

- For separation along one x^a between #1 and #2, i.e. for $\underline{p=d-2}$ (e.g. D8 in 10 dimensions) it's clear which it must be: x^a.

But what if $\underline{p<d-2}$? Then there is more than one state in the set

$$(a_1^a)^\dagger \, |p^+, \vec{p}_T \,;\, [12]\rangle .$$

- For this case, the right linear combination is

(5.1) $$\boxed{\sum_a (\bar{x}_2^a - \bar{x}_1^a)(a_1^a)^\dagger \, |p^+, \vec{p}_T \,;\, [12]\rangle}$$

which is the only combo that treats all (a) directions the same, and hence must be right one

Yang-Mills on the worldvolume

Notice the fascinating dependence on $(\bar{x}_2^a - \bar{x}_1^a)$:

For the 1-oscillator states (all of them) we have

(5.2) $$M^2 = \left|\frac{(\bar{x}_2^a - \bar{x}_1^a)}{2\pi\alpha'}\right|^2 \geq 0$$

$$= 0 \quad \text{when} \quad \bar{x}_2^a = \bar{x}_1^a \quad , \; a \perp \text{Dp-brane worldvolume}$$

$\Leftrightarrow$ coincident D-branes possess N^2 massless gauge fields and scalars living on worldvolume

$\begin{cases} \text{first endpoint (oriented strings) on Dp-brane } i \\ \text{second " " " " " } j \end{cases}$

where $i, j = 1, \ldots, N$.

The theory can actually be shown to possess gauge group $U(N)$.

(Review: $A_\mu = A_\mu^a T^a$; $A_{(1)} = A_\mu dx^\mu$
↑ generators in Lie algebra g

Covariant derivative
$$D_\mu = \partial_\mu + [A_\mu, \ ]$$

$$F_{(2)} = dA_{(1)} + \underbrace{A_{(1)} \wedge A_{(1)}}$$

nonzero because $[T^a, T^b] \neq 0$; in fact $[T^a, T^b] = i f^{ab}{}_c T^c$
↑ structure constants of Lie group G.

Bianchi identity
$$D \wedge F = 0$$

Group element
$$U = \exp(i w^A T_A)$$
↑ additive parameters (like rotation or rapidity BUT $(T^A)^a{}_b$ act only on "internal" indices a on fields

Some field $\{q^a\}$ gets acted on as
$$q \to q' = Uq$$
by an element of the group G.
Then a utility of D is that
$$Dq \to U(Dq)$$
and
$$F \to UFU^{-1}$$
while
$$A \to UAU^{-1} + (dU)U^{-1}$$

Action principle
$$S_g = \int d^D x \sqrt{-g} \left(-\frac{1}{4} \mathrm{Tr} \left(F^{\mu\nu} F_{\mu\nu} \right) \right)$$
and matter couples via the co<u>vari</u>ant derivative.

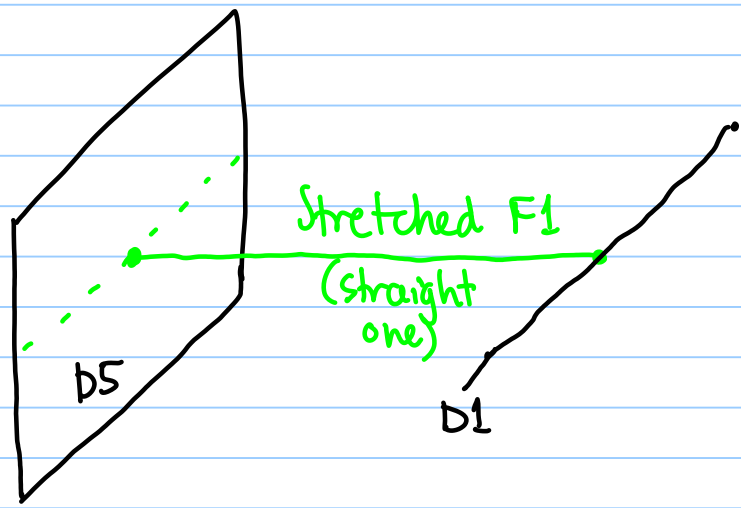
<u>Note</u>: open-strings carry non-dynamical* gauge labels $\underline{N}$ (Chan-Paton factors) to distinguish between D-branes

*(no energy cost)

Strings between $Dp \,//\, Dq$

We don't want to be stuck with an ability to analyze only string states for separated but otherwise identical Dp-branes ! ☺

$\Rightarrow$ consider setup of $Dp \,//\, Dq$, e.g. $D1 \perp D5$:



For 1 Dp & 1 Dq we still have four sectors of stretched strings, but now they are (p,p), (q,q), (p,q) & (q,p) strings (!)

We break up our coords into 3 classes :- $(p>q)$

(7.1)
Common tangential : $x^0, x^1, \ldots x^q$ (NN)
mixed : $x^{q+1}, x^{q+2}, \ldots, x^p$ (ND)
common normal : $x^{p+1}, \ldots, x^{d-1}$ (DD)

In light-cone gauge, Zwiebach labels these as

(7.2) $$\underbrace{x^+, x^-, \{x^i\}}_{NN} \;;\; \underbrace{\{x^r\}}_{ND} \;;\; \underbrace{\{x^a\}}_{DD}$$

- Since we already know how to quantize NN & DD X fields, let's now concentrate on the new, ND, face:-

ND modes

These are the mixed coords $X^r(\tau,\sigma)$ and satisfy BCs

(7.3) $$X^{r\prime}(\tau,\sigma)\Big|_{\sigma=0} = 0 \quad ; \quad X^r(\tau,\sigma)\Big|_{\sigma=\pi} = \bar{x}^r_2 \,.$$

We can write, as usual, for wave eqn soln

(8.1) $$X^r(\tau,\sigma) = \frac{1}{2}\left(f^r(\tau+\sigma) + g^r(\tau-\sigma)\right)$$

Imposing N condition at $\sigma=0$ gives

$$f^{r\prime}(u) = g^{r\prime}(u), \Rightarrow g^r(u) = f^r(u) + \underbrace{c_0^r}_{\text{choose } \equiv 2\bar{x}_2^r}$$

(8.2) so $$X^r(\tau,\sigma) = \bar{x}_2^r + \frac{1}{2}\left(f^r(\tau+\sigma) + f^r(\tau-\sigma)\right)$$

Imposing D condition at $\sigma=\pi$ gives, directly,

(8.3) $$f^r(u+2\pi) = -f^r(u)$$

∴ Makes sense to mode-expand with periodicity 4π this time, i.e.

(8.4) $$f^r(u) = \sum_{n=0}^{\infty}\left[f_n^r \cos\left(\frac{nu}{2}\right) + h_n^r \sin\left(\frac{nu}{2}\right)\right]$$

For antiperiodicity to hold, we sum only over odd n, not even n. (In particular, no ∅ mode included!)

(8.5) $$\Rightarrow X^r(\tau,\sigma) = \bar{x}_2^r + \sum_{n\,\mathrm{odd}}\left[A_n^r \cos\left(\frac{n\tau}{2}\right) + B_n^r \sin\left(\frac{n\tau}{2}\right)\right]\cos\left(\frac{n\sigma}{2}\right)$$

Since we have antiperiodicity, we should actually expand in half-odd-integrally moded oscillators

(8.6) $$\boxed{X^r(\tau,\sigma) = \bar{x}_2^r + i\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}_{odd}}\frac{2}{n}\,\alpha_{n/2}^r\, e^{-i(n/2)\tau}\cos\left(\frac{n\sigma}{2}\right) \quad (ND)}$$

(again, needed to have $(\alpha_{n/2}^r)^\dagger = \alpha_{-n/2}^r \ldots$)

This makes sure, among other things, that

(8.7) $$\dot{X}^r \pm X^{r\prime} = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}_{odd}}\alpha_{n/2}^r\, e^{-i(n/2)(\tau\pm\sigma)}$$

Then the $\{\alpha_{n/2}^r,\ n\in\mathbb{Z}_{odd}\}$ can be shown to obey the standard CCR's starting from

(8.8) $$[X^r(\tau,\sigma), \dot{X}^s(\tau,\sigma')] = i(2\pi\alpha')\,\delta(\sigma-\sigma')\,\delta^{rs};$$

Z pp292-3 shows explicitly that this gives

(9.1) $$[\alpha^r_{n/2}, \alpha^s_{m/2}] = \frac{n}{2}\delta^{rs}\delta_{n+m,0}$$ (✓)

Mass-squared operator

As you might imagine, M^2 gets contributions from all three sectors — NN, ND and DD — because that's what it takes to cover $X^0, \ldots, X^{d-1}$ ← critical dimension

By analogy,

(9.2) $$2p^+p^- = \frac{1}{\alpha'}\Big[\alpha' p^i p^i + \frac{1}{2}\alpha^a_0\alpha^a_0 + \sum_{n=1}^{\infty}(\alpha^i_{-n}\alpha^i_n + \alpha^a_{-n}\alpha^a_n) + \sum_{m\in\mathbb{Z}^+_{odd}}\alpha^r_{-m/2}\alpha^r_{m/2} + a\Big]$$

OK: The question is: what is the normal ordering constant, a ?

★ We had that each integrally-moded $X^\perp$ field contributed $(-\frac{1}{24})$ to a: $\frac{1}{2}\left(\sum_n n\right) = \frac{1}{2}\left(-\frac{1}{12}\right) = -\frac{1}{24}$

So for each NN and each DD coord we get $-\frac{1}{24}$:

(9.3) $$\boxed{a_{NN} = -\frac{1}{24} \quad a_{DD} = -\frac{1}{24}}$$

★ What about the half-integrally-moded ND $\{X^r\}$?

We are asking about

(9.4) $$\frac{1}{2}\sum_{m\in\mathbb{Z}_{odd}}\alpha^r_{-m/2}\alpha^r_{m/2} = \sum_{m\in\mathbb{Z}^+_{odd}}\alpha^r_{-m/2}\alpha^r_{m/2} + \frac{1}{2}\sum_{m\in\mathbb{Z}^+_{odd}}\left[\alpha^r_{\frac{m}{2}}, \alpha^r_{-\frac{m}{2}}\right]$$

- There are how many ND coords? Well, we have $(q+1)$ NN's $(q<p)$ and $(d-1)-p = d-(p+1)$ DD's

so there are $d-(q+1)-[d-(p+1)]$ ND's, i.e.

(10.1) $$\boxed{\#_{ND} = p-q}$$

Then

(10.2) $$a_{ND} = \frac{1}{2}(p-q)\sum_{m\in\mathbb{Z}^+_{odd}} \frac{m}{2} = \frac{(p-q)}{4}\Big(\sum_{m\in\mathbb{Z}^+_{odd}} m\Big)$$

How do we find $\Big(\sum_{m\in\mathbb{Z}^+_{odd}} m\Big)$?

Well, $$\sum_{m\in\mathbb{Z}^+_{odd}} m + \sum_{m\in\mathbb{Z}^+_{even}} m = \sum_{m\in\mathbb{Z}^+} m = \zeta(-1)$$

$$= \sum_{m\in\mathbb{Z}^+_{odd}} m + 2\sum_{m\in\mathbb{Z}^+} m$$

So $$\sum_{m\in\mathbb{Z}^+_{odd}} m = \zeta(-1) - 2\zeta(-1) = -\zeta(-1) = +\frac{1}{12}$$

(10.3) $$\Rightarrow \boxed{a_{ND} = +\frac{1}{48}}$$ and $(p-q)$ of these.

Now we can actually compute the total normal-ordering constant for our Dp // Dq system:

$$a_{Dp//Dq} = (p+1)\left(\frac{-1}{24}\right) + (d-(p+1))\left(\frac{-1}{24}\right) + (p-q)\left(\frac{+1}{48}\right)$$

$$= -\frac{1}{24}\big(d-(p-q)\big) + \frac{(p-q)}{48}$$

(10.4) $$= -1 + \frac{(p-q)}{16}$$

$$\Rightarrow M^2 = \left|\frac{(x_2^a - x_1^a)}{2\pi\alpha'}\right|^2 + \frac{1}{\alpha'}\left[N^\perp - 1 + \frac{(p-q)}{16}\right]$$

where

(10.5) $$N^\perp = \sum_{n=1}^{\infty}\sum_{i=2}^{q} n(a_n^i)^\dagger(a_n^i) + \sum_{m=1}^{\infty}\sum_{a=p+1}^{d-1} m\,(a_m^a)^\dagger(a_m^a) + \sum_{k\in\mathbb{Z}^+_{odd}}\sum_{r=q+1}^{p}\frac{k}{2}(a^r_{k/2})^\dagger(a^r_{k/2})$$

Dp//Dq