ST 24 Sep 04

Note Title 24/09/2004

The σ coördinate

Last time, we derived the classical equations of motion for the relativistic string. In part II, we set the static gauge

"target" time $\rightarrow$ $x^0 \equiv ct = c\tau$ $\leftarrow$ "worldsheet" time

but we let the other spatial coord be "whatever". Today we will analyze how much simpler the physics appears when we make a special choice spatially too.

Choose σ and τ to be orthogonal:

(1.1) $$\boxed{\frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial \sigma} = 0}$$

This holds everywhere along the string, not just at the endpoints!

Since ∂_σ points along the string, $\frac{\partial \vec{X}}{\partial t}$ is $\perp$ to string,

(1.2) hence is just the velocity : $\boxed{\vec{v}_\perp = \frac{\partial \vec{X}}{\partial t}}$ (3-vector)

Then our previous, rather nasty, expressions collapse to $\dot{X}\cdot X' = 0$ and $\dot{X}^2 = -c^2 + \vec{v}_\perp^2$ while $(X')^2 = \left|\frac{\partial \vec{X}}{\partial \sigma}\right|^2$

and so in this special worldsheet-spatial coord,

$(X')^2 = \left(\frac{ds}{d\sigma}\right)^2$ where ds = infinitesimal spatial arc length

by definition of the arc length.

(1.3) Before, we had e-o-m $\frac{\partial \mathcal{P}^\tau{}_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma{}_\mu}{\partial \sigma} = 0$

and boundary conditions : $\begin{cases} \text{either Dirichlet or} \\ \text{no momentum flow off ends.} \end{cases}$

Let's see the simplifications in this "σ parametrization".

From before,

$$(2.1a)\quad \boxed{\mathcal{P}^{\tau}{}_{\mu} = -\frac{T_0}{c}\,\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2(X')^2}}}$$

and similarly

$$(2.1b)\quad \boxed{\mathcal{P}^{\sigma}{}_{\mu} = -\frac{T_0}{c}\,\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2(X')^2}}}$$

Now these collapse to

$$(2.2a)\quad \mathcal{P}^{\tau\mu} = -\frac{T_0}{c}\left(-\left(\frac{ds}{d\sigma}\right)^2\right)\Big/\left((c^2-\vec{v}_\perp^{\,2})\left(\frac{ds}{d\sigma}\right)^2\right)^{1/2}\frac{\partial X^\mu}{\partial t} = \frac{T_0}{c^2}\,\frac{ds}{d\sigma}\,\frac{(\partial X^\mu/\partial t)}{\sqrt{1-\vec{v}_\perp^{\,2}/c^2}}$$

$$(2.2b)\quad \mathcal{P}^{\sigma\mu} = -\frac{T_0}{c}\,(c^2-\vec{v}_\perp^{\,2})\,\frac{\partial X^\mu}{\partial\sigma}\Big/\left[\frac{ds}{d\sigma}\sqrt{c^2-\vec{v}_\perp^{\,2}}\right] = -T_0\,\frac{\partial X^\mu}{\partial s}\sqrt{1-\vec{v}_\perp^{\,2}/c^2}$$

The e-o-m collapses to

$$\frac{T_0}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial X^\mu}{\partial t}\,\frac{ds}{d\sigma}\,\frac{1}{\sqrt{1-\vec{v}_\perp^{\,2}/c^2}}\right) = T_0\frac{\partial}{\partial\sigma}\left(\frac{\partial X^\mu}{\partial s}\sqrt{1-\vec{v}_\perp^{\,2}/c^2}\right)$$

$\underline{\mu=0}$
$$\frac{T_0}{c}\frac{\partial}{\partial t}\left(\frac{ds}{d\sigma}\,\frac{1}{\sqrt{1-\vec{v}_\perp^{\,2}/c^2}}\right) = \partial_\tau\mathcal{P}^{\tau 0} = \partial_\sigma\mathcal{P}^{\sigma 0} = 0$$

$$(2.3)\quad \text{or}\quad \frac{\partial}{\partial t}\left(\frac{(T_0/c)}{\sqrt{1-\vec{v}_\perp^{\,2}/c^2}}\,\frac{ds}{d\sigma}\right) = 0 \quad (*)$$

In an infinitesimal piece of string, this says 0th cpt of 4-mom., $\dfrac{(T_0/c)(ds/d\sigma)}{\sqrt{1-\vec{v}_\perp^{\,2}/c^2}}$ is conserved.

$$(2.4)\quad \Rightarrow\quad H = \int ds\,\frac{T_0}{\sqrt{1-\vec{v}_\perp^{\,2}/c^2}}$$

$\longrightarrow$ rest tension of ∞esimal segment of string

$\mu = i\ \ (i=1,2,3)$

$$\partial_\tau\mathcal{P}^{\tau i} = \frac{T_0}{c^2}\frac{\partial}{\partial t}\left(\frac{v_\perp^i}{\sqrt{1-v_\perp^2/c^2}}\frac{ds}{d\sigma}\right) = \frac{T_0}{c^2}\left(\frac{ds}{d\sigma}\right)\frac{\partial}{\partial t}\left(\frac{v_\perp^i}{\sqrt{1-\vec{v}_\perp^{\,2}/c^2}}\right)$$

whereas

$$\partial_\sigma\mathcal{P}^{\sigma i} = -T_0\,\frac{ds}{d\sigma}\,\frac{\partial}{\partial s}\left(\frac{\partial X^\mu}{\partial s}\sqrt{1-\vec{v}_\perp^{\,2}/c^2}\right)\quad \text{so that}$$

$$(2.5)\quad \text{e-om is}\quad \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{v_\perp^i}{\sqrt{1-\vec{v}_\perp^{\,2}/c^2}}\right) = \frac{\partial}{\partial s}\left(\frac{\partial X^i}{\partial s}\sqrt{1-\vec{v}_\perp^{\,2}/c^2}\right).$$

Use the other eqn $*$ to effect: we have the simplification $\dfrac{\partial}{\partial t}\dfrac{v_\perp^i}{\sqrt{1-v_\perp^2/c^2}} = \dfrac{1}{\sqrt{---}}\dfrac{\partial v_\perp^i}{\partial t}$.

So: EOM becomes

(3.1) $$\frac{\partial}{\partial s}\left(\frac{\partial X^i}{\partial s}\sqrt{1-\vec{v}_\perp^{\,2}/c^2}\right) = \frac{1}{c^2}\frac{\partial^2 X^i}{\partial t^2}\frac{1}{\sqrt{1-\vec{v}_\perp^{\,2}/c^2}}$$

rather like a non-relativistic string with energy density $T_{eff} = T_0\sqrt{1-\frac{\vec{v}_\perp^{\,2}}{c^2}}$, $\mu_{eff} = \frac{(T_0/c^2)}{\sqrt{1-\vec{v}_\perp^{\,2}/c^2}}$

Assume equal length bits $d\sigma$ of string carry the same energy. Then

(3.2) $$\frac{1}{c^2}\frac{\partial^2\vec{X}}{\partial t^2} = \frac{\sqrt{1-\vec{v}_\perp^{\,2}/c^2}}{(ds/d\sigma)}\,\partial_\sigma\left[\frac{\sqrt{1-\vec{v}_\perp^{\,2}/c^2}}{(ds/d\sigma)}\frac{\partial\vec{X}}{\partial\sigma}\right]$$

call this $A(\sigma)^{-1}$

Then
$$\frac{1}{c^2}\frac{\partial^2\vec{X}}{\partial t^2} = \frac{1}{A(\sigma)}\frac{\partial}{\partial\sigma}\left[A(\sigma)\frac{\partial\vec{X}}{\partial\sigma}\right]$$

This eqn begs us to define σ' by
$$\frac{\partial}{\partial\sigma'} = \frac{1}{A(\sigma)}\frac{\partial}{\partial\sigma} \Rightarrow$$

(3.3) $$\boxed{\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial(\sigma')^2}\right]X(\tau,\sigma) = 0}$$ Wave Eqn

Endpoint conditions

(3.4) Dirichlet BC's say: $\boxed{\left.\frac{\partial X}{\partial\tau}\right|_{\text{endpts}} = 0}$ D

For the free-boundary condition we wrote in terms of $\mathcal{P}^\sigma_\mu$
$$\mathcal{P}^\sigma_\mu = 0 \quad @ \text{ endpts}$$
Very nicely, this collapses in σ vrble (in static gauge)
$\Rightarrow$ compute
$$\vec{\mathcal{P}}^\sigma = -T_0\frac{\partial\vec{X}}{\partial s}\sqrt{1-\vec{v}_\perp^{\,2}/c^2}$$

so the free-endpt BC simplifies to

(3.5) $$\boxed{\left.\frac{\partial\vec{X}}{\partial\sigma}\right|_{\text{endpts}} = 0}$$ N

The constraint $\left(\frac{\partial\vec{X}}{\partial\sigma}\right)^2 + \frac{1}{c^2}\left(\frac{\partial\vec{X}}{\partial t}\right)^2 = 0$

That $A(\sigma)$ "mystery" function has another role:

(4.1) $$d\sigma' = \frac{ds}{\sqrt{1-\vec{v}_\perp^2/c^2}} = \frac{1}{T_0}\, dE$$

$\Rightarrow$ in σ' parametrization, $dE = T_0\, d\sigma'$ i.e. tension is constant.

- **Q orthogonality in σ':-**

Since we know $\frac{\partial \vec{X}}{\partial \tau} \cdot \frac{\partial \vec{X}}{\partial \sigma} = 0$,

$$= \frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial \sigma'} \left(\frac{d\sigma'}{d\sigma}\right) = 0 \quad (\checkmark)$$

i.e. $\frac{\partial \vec{X}}{\partial \tau} \cdot \frac{\partial \vec{X}}{\partial \sigma'} = 0$ still orthogonal ☺

Z§7-4 Solving the e-o-m for a general open string :-

- Use the method of characteristics (for example) $\Rightarrow$

(4.2) $$\vec{X}(\tau,\sigma) = \frac{1}{2}\left[\vec{F}(ct+\sigma) + \vec{G}(ct-\sigma)\right]$$

(orthogonality condition $\frac{\partial \vec{X}}{\partial t} \cdot \frac{\partial \vec{X}}{\partial \sigma} = 0$ and

Add e-o-m $\frac{1}{c^2}\left(\frac{\partial \vec{X}}{\partial t}\right)^2 + \left(\frac{\partial \vec{X}}{\partial \sigma}\right)^2 = 0$

(4.3) $\Rightarrow \frac{\partial \vec{X}}{\partial \sigma} = \pm \frac{1}{c}\frac{\partial \vec{X}}{\partial t}$. $\Rightarrow$ can write eqn (4.2) ☺)

- Then Neumann (free endpt) BC's relate L-movers ($\vec{F}$) and R-movers ($\vec{G}$) $\Rightarrow$ for simplicity use only $\vec{F}$. Absorb arb. const. from integration into $\vec{F}$ $\Rightarrow$

(4.4) $$\boxed{\vec{X}(t,\sigma) = \frac{1}{2}\left[\vec{F}(ct+\sigma) + \vec{F}(ct-\sigma)\right]}$$

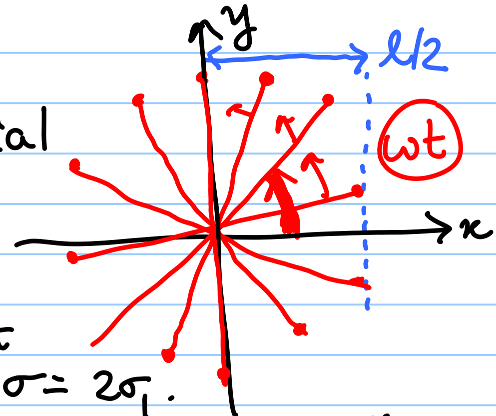
Combining this with (4.3) gives

(4.5) $$\boxed{\left|\frac{d\vec{F}}{du}\right|^2 = 1}$$

$\Rightarrow$ can think of u as length parameter along curve, and $F(u)$ as the position of the $\sigma=0$ endpt @ time u/c.

More details in Zwiebach §7.4, pp 126-7

Rigid rotator



- Can get some nontrivial info just by knowing how the endpoints move (☺).
  Let one endpt be at $\sigma=0$, the other at $\sigma=2\sigma_1$.
  The $\sigma=0$ endpoint we take to be the one moving as

(5.1) $$\vec{X}(t,0) = \frac{\ell}{2}\{\cos\omega t, \sin\omega t\} = \vec{F}(ct)$$

- The centre of mass of this string doesn't move, so $\vec{F}(\sigma+2\sigma_1)=\vec{F}(\sigma)$ (otherwise it would be quasi-periodic with $\vec{F}(\sigma+2\sigma_1)=\vec{F}(\sigma)+2\sigma_1\cdot\frac{\vec{v}_0}{c}$ where $\vec{v}_0 =$ c.o.m velocity

Periodicity $\Rightarrow \vec{F}(u)=\frac{\ell}{2}\{\cos\left(\frac{\omega u}{c}\right), \sin\left(\frac{\omega u}{c}\right)\}$ where

(5.2) $\frac{\omega}{c}\cdot 2\sigma_1 = 2\pi$ (this choice 2π, rather than $2\pi m$ for $m\in\mathbb{Z}$, means our parametrization doesn't backtrack)

- Endpts move @ c ($\Leftrightarrow |d\vec{F}/du|^2=1$) so that
  $$\left(\frac{\omega\ell}{c\cdot 2}\right)^2=1$$

(5.3) i.e. $\ell = \frac{2c}{\omega} = \frac{2\sigma_1}{\pi}$. But $\sigma_1 = E/T_0$, so $\ell = \frac{2}{\pi}\frac{E}{T_0}$.

▷ This string is shorter (!) than a static string @ energy E; this is because some energy E had to go into the kinetic energy of rotation.

▷ Note also that $E = T_0(\pi c/\omega)$ i.e. $E \propto 1/\omega$; this "odd" behaviour again originates in the requirement that the endpts have to go at the speed of light. Pumping in energy, by (5.1), makes the string longer, but those endpts still have to go @ c - no matter what - so less can go into rotational kinetic energy.

- Lastly, algebra

(5.4) $$\Rightarrow \underbrace{\vec{X}(t,\sigma) = \frac{\sigma_1}{\pi}\overbrace{\cos\left(\frac{\pi\sigma}{\sigma_1}\right)}^{\text{standing wave}}\overbrace{\left\{\cos\left(\frac{\pi ct}{\sigma_1}\right), \sin\left(\frac{\pi ct}{\sigma_1}\right)\right\}}^{\text{circles around.}}}$$

World-Sheet Currents ☺

- It is a general principle that symmetry in physics gives rise to conservation laws. For continuous symmetry [Lie groups] this can be codified precisely in Noether's Theorem..(which some of you know quite well already while others don't, yet.)
- Writing Maxwell's equations using 4-vector potential

(6.1) A_μ via $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the story goes

(6.2) $$\partial_\mu F^{\mu\nu} = J^\nu \qquad \{J^\nu\} = (c\rho, \vec{J})$$

$$\Rightarrow \partial_\nu(\partial_\mu F^{\mu\nu}) = \partial_\nu J^\nu$$

(6.3) $$= \partial_{(\nu}\partial_{\mu)} F^{[\mu\nu]} \equiv 0 \qquad \text{i.e. } \partial_\nu J^\nu = 0$$ "J is conserved"

(6.4) So $\frac{1}{c}\partial_t(c\rho) + \vec{\nabla}\cdot\vec{j} = 0$

(6.5) $Q(t) = \int d^3x\, \rho$ is conserved because

(6.6) $$\frac{dQ}{dt} = \int d^3x \frac{\partial\rho}{\partial t} = -\int d^3x\, \vec{\nabla}\cdot\vec{j} = -\oint_S \vec{j}\cdot d\vec{A}$$

As $S \to$ infinitely far away, $\frac{dQ}{dt} \to 0$.

<u>Next</u>: Noether's Theorem ♀

Lagrangian methods

Consider $q(t) \to q(t) + \delta q(t)$.

Induces $\dot{q}(t) \to \dot{q}(t) + \delta\dot{q}(t)$

If $\delta S = 0$ under δq, then

(6.7) define $$\epsilon Q \equiv \frac{\partial L}{\partial \dot{q}}\delta q$$

and $\frac{dQ}{dt} = 0$ can be proven from e-o-m's

(7.1) EoM's $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) - \frac{\partial L}{\partial q^a} = 0$

Symmetry: $\frac{\partial L}{\partial q^a}\delta q^a + \frac{\partial L}{\partial \dot{q}^a}\delta \dot{q}^a = 0$

So $\epsilon \frac{dQ}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right)\delta q^a + \frac{\partial L}{\partial \dot{q}^a}\frac{d}{dt}(\delta q^a)$

(7.2) $= \frac{\partial L}{\partial q^a}\delta q^a + \frac{\partial L}{\partial \dot{q}^a}\delta \dot{q}^a = 0$ ∎

Example: Space translation

(7.3) $q^a \to q^a + \epsilon^a$ ← (const)

then $\dot{q}^a \to \dot{q}^a$

$\mathcal{L}$ depends (for a free string/ particle) only on speed and not explicitly on time, so

$\epsilon Q = \frac{\partial \mathcal{L}}{\partial \dot{q}^a}\delta q^a = \frac{\partial L}{\partial \dot{q}^a}\epsilon \Rightarrow Q = \frac{\partial L}{\partial \dot{q}^a} = p_a$.

Time translation symmetry ⟺ momentum p_a ← assoc. w. q^a. conservation.

D New(er) concept: fields (e.g. $X^\mu(\tau,\sigma)$!)

Do Lagrangians à la

(7.4)
$$\begin{cases} t \to x^\mu \\ q^a(t) \to \phi^a(x^\mu) \\ p_a{}^\mu \to \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} \end{cases}$$

General ∞-esimal variation can be written as

$$\delta\phi^a = \epsilon^\alpha h_i^a(\phi)$$

If $\mathcal{L}$ is invariant under this, then

(7.5) $$\epsilon^i j_i^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}\delta\phi^a$$

are conserved currents: $\partial_\alpha j_i^\alpha = 0$ by virtue of the E-L eqns.

Worldsheet Currents

We have a 2-d field theory living on our string worldsheet: the fields (maps) are

$X^\mu(\tau,\sigma)$: worldsheet $\to$ target space

Following Zwiebach lets call worldsheet coords

$(\xi^0, \xi^1) = (\tau, \sigma)$

and our action is

(8.1) $$S = \int d\xi^0 d\xi^1 \mathcal{L}(\partial_0 X^\mu, \partial_1 X^\nu) \qquad \left(\partial_\alpha = \frac{\partial}{\partial \xi^\alpha}\right)$$

Fields are string coords.

Want conserved current. Need field variation that doesn't change $\mathcal{L}$.

(8.2) Try $\delta X^\mu = \epsilon^\mu$ (constant D-vector) spacetime translations

Note that

(8.3) $$\partial_\alpha(\delta X^\mu) = \partial_\alpha \epsilon^\mu = 0$$
$$= \delta(\partial_\alpha X^\mu)$$

(8.4) Noether current $$\epsilon^\mu j^\alpha_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} \delta X^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\alpha X^\mu} \epsilon^\mu$$

so $$j^\alpha_\mu = \frac{\partial \mathcal{L}}{\partial \partial_\alpha X^\mu}$$

(8.5) i.e. $$\{j^0_\mu, j^1_\mu\} = \left\{\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}, \frac{\partial \mathcal{L}}{\partial X'^\mu}\right\} = \boxed{\{\mathcal{P}^\tau_\mu, \mathcal{P}^\sigma_\mu\}}$$

The current conservation eqn is just the string worldsheet eqn of motion

$$\partial_\tau \mathcal{P}^\tau_\mu + \partial_\sigma \mathcal{P}^\sigma_\mu = 0.$$

(8.6) Define momentum: $$p_\mu = \int_0^{\sigma_1} d\sigma\, \mathcal{P}^\tau_\mu(\sigma)\Big|_{\tau \text{ fixed}}$$

i.e. $\mathcal{P}^\tau_\mu$ is the spacetime momentum density carried by the string

$$\frac{dp_\mu}{d\tau} = \int_0^{\sigma_1} d\sigma \frac{\partial \mathcal{P}^\tau{}_\mu}{\partial \tau} = \int_0^{\sigma_1} d\sigma \left(-\frac{\partial \mathcal{P}^\sigma{}_\mu}{\partial \sigma}\right) = \mathcal{P}^\sigma{}_\mu \Big]_0^{\sigma_1}$$

$= 0$ for closed string by continuity: $\sigma_1 \in [0, \sigma_1]$

$= 0$ for open string by free-ends BC.

(9.1) $\boxed{\frac{dp_\mu}{d\tau} = 0}$ is a worldsheet conservation equation.

It has an interpretation in static gauge where it also says $\frac{dp_\mu}{dt} = 0$ ← (coord time)

Now, $p_\mu = \int_0^{\sigma_1} d\sigma\, \mathcal{P}^\sigma{}_\mu$ actually catches the flux of $\mathcal{P}^\sigma{}_\mu$ across the curve integrated over. $\mathcal{P}^\tau{}_\mu$, on the other hand, is tangent.

So in fact, flux $= \{\mathcal{P}^\tau_\mu, \mathcal{P}^\sigma_\mu\} \cdot \underbrace{\begin{Bmatrix} d\sigma \\ -d\tau \end{Bmatrix}}_{\text{outgoing normal vector}} = \mathcal{P}^\tau_\mu\, d\sigma - \mathcal{P}^\sigma_\mu\, d\tau$

Flux across curve Γ

(9.2) $p_\mu = \int_\Gamma \mathcal{P}^\tau{}_\mu\, d\sigma - \mathcal{P}^\sigma{}_\mu\, d\tau$

$$\underset{\text{Stokes' thm}}{=} \int_R \left(\frac{\partial \mathcal{P}^\tau{}_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma{}_\mu}{\partial \sigma}\right) d\tau\, d\sigma = 0 \quad \text{by e.o.m.}$$

The integral is actually independent of the path chosen here,

$\boxed{\text{Lorentz Currents}}$

Exercise:
Figure this out for yourself, e.g. for ∞esimal rotations.

(9.3) $\delta X^\mu = \epsilon^{\mu\nu} X_\nu$ where $\epsilon^{\mu\nu}$ is antisymmetric.

We can guess the form of ϵ ↑ by demanding that $\delta(X^2) = (2\eta_{\mu\nu} X^\mu \delta X^\nu) = 2\eta_{\mu\nu} X^\mu \epsilon^{\nu\lambda} X_\lambda$

$= 2\eta_{\mu\nu} X^\mu X^\lambda \epsilon_{\nu\lambda}$

$= 0$ by symmetry/antisymmetry

$\epsilon^{\mu\nu}$ also needs to be antisymmetric so it has only $\frac{4 \cdot 3}{2!} = 6$ indep cpts: 3 rotations & 3 boosts

$\mathcal{L}$ for string involves $\frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta} \eta_{\mu\nu}$

(10.1) $$\delta\left(\eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b}\right) = \eta_{\mu\nu}\, \delta\left(\frac{\partial X^\mu}{\partial \xi^a}\right) \frac{\partial X^\nu}{\partial \xi^b} + {\scriptstyle \mu \leftrightarrow \nu \atop a \leftrightarrow b}$$

$$= \eta_{\mu\nu} \frac{\partial\, \delta X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} + {\scriptstyle a \leftrightarrow b \atop \mu \leftrightarrow \nu}$$

$$= \eta_{\mu\nu}\left(\epsilon^{\mu\rho} \frac{\partial X_\rho}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} + \frac{\partial X^\mu}{\partial \xi^a} \epsilon^{\nu\lambda} \frac{\partial X_\lambda}{\partial \xi^b}\right)$$

$$= \epsilon^{\mu\rho} \frac{\partial X_\rho}{\partial \xi^a} \frac{\partial X_\mu}{\partial \xi^b} + \epsilon^{\nu\lambda} \frac{\partial X_\nu}{\partial \xi^a} \frac{\partial X_\lambda}{\partial \xi^b}$$

(10.2) $$= \epsilon_{\alpha\beta} \frac{\partial X^\beta}{\partial \xi^a} \frac{\partial X^\alpha}{\partial \xi^b} + \epsilon_{\alpha\beta} \frac{\partial X^\alpha}{\partial \xi^a} \frac{\partial X^\beta}{\partial \xi^b}.$$

($\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$ needed.)

Then
$$\epsilon^{\mu\nu} j^a_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_a X^\mu} \delta X^\mu = \mathcal{P}^a_\mu \epsilon^{\mu\nu} X_\nu$$

$$\Rightarrow \epsilon^{\mu\nu} j^a_{\mu\nu} = -\tfrac{1}{2} \epsilon^{\mu\nu} \left(X_\mu \mathcal{P}^\alpha_\nu - X_\nu \mathcal{P}^\alpha_\mu\right)$$

(10.3) $\Rightarrow$ currents $\boxed{\mathcal{M}^\alpha_{\mu\nu} = X_\mu \mathcal{P}^\alpha{}_\nu - X_\nu \mathcal{P}^\alpha{}_\mu}$ (are antisym.)

Current conservation $\Rightarrow$ $\partial_\tau \mathcal{M}^\tau_{\mu\nu} + \partial_\sigma \mathcal{M}^\sigma_{\mu\nu} = 0$ so

Charges: conserved ω ∂_t:

(10.4) $$M_{\mu\nu} = \int_\gamma \left(d\sigma\, \mathcal{M}^\tau_{\mu\nu} - d\tau\, \mathcal{M}^\sigma_{\mu\nu}\right)$$

M_{ij} are associated with boosts and rotations.

Ex: Calculate them for a string configuration of interest - e.g. the rigid rotator.

Identifying conserved quantum #s.

Let's go back and think about a rigid "2x4"! ☺
Let it have length L, and mass m. Let's find energy E and angular momentum J.
We have

$$E = \tfrac{1}{2} I \omega^2$$

Moment of inertia $\displaystyle I = \int r^2 dm = \int_{-L/2}^{L/2} dl\, \mu\, l^2$

$$= \mu \left.\frac{l^3}{3}\right]_{-L/2}^{L/2} = \mu\left(\frac{L^3}{24} + \frac{L^3}{24}\right) = \mu \frac{L^3}{12} = \frac{mL^2}{12}$$

(I think ☺)

For the angular momentum,

$$J = I\omega$$
$$= \frac{mL^2}{12}\omega .$$

For the energy

$$E = \frac{1}{2}\frac{mL^2\omega^2}{12} \Rightarrow \left(\frac{24E}{mL^2}\right)^{1/2} = \omega$$

so

$$J = \frac{mL^2}{12}\sqrt{\frac{24E}{mL^2}} = \sqrt{E}\cdot\sqrt{\frac{mL^2}{6}}$$

(11.1) i.e. $\boxed{J = \sqrt{\frac{mL^2}{6}}\sqrt{E}}$

Now let's look @ strings, even the rigid rotators:-

Angular momentum in the (12) plane is given by

(11.2) $$M_{12} = \int d\sigma \left(X_1 \mathcal{P}^\tau_2 - X_2 \mathcal{P}^\tau_1\right)$$

(11.3) Rotator: $$\vec{X}(t,\sigma) = \frac{\sigma_1}{\pi}\cos\frac{\pi\sigma}{\sigma_1}\left\{\cos\left(\frac{\pi ct}{\sigma_1}\right), \sin\left(\frac{\pi ct}{\sigma_1}\right)\right\}$$

(11.4) so $$\vec{\mathcal{P}}^\tau = \frac{T_0}{c^2}\frac{\partial \vec{X}}{\partial t} = \frac{T_0}{c}\cos\left(\frac{\pi\sigma}{\sigma_1}\right)\left\{-\sin\left(\frac{\pi ct}{\sigma_1}\right), +\cos\left(\frac{\pi ct}{\sigma_1}\right)\right\}$$

(11.5) so that $$M_{12} = \int_0^{\sigma_1} d\sigma \left(\cos^2\left(\frac{\pi\sigma}{\sigma_1}\right)\right)\left[\frac{\sigma_1}{\pi}\frac{T_0}{c}\right] = \frac{\sigma_1^2 T_0}{2\pi c} = |J|$$

Then energy is proportional to tension:

(12-1) $$E = T_0 \cdot \sigma_1$$

(12-2) So $$|J| = E^2 \frac{1}{2\pi c T_0}$$

(12-3) Let $$\boxed{\alpha' = \frac{1}{2\pi T_0 \hbar c}}$$ i.e. $$\boxed{T_0 = \frac{1}{2\pi \alpha' \hbar c}}$$

▷ α' is the slope of lines of $J/\hbar$ plotted against energy-squared. (Regge trajectories.) The stone-age string theorists worked out a fairly good match between this behavior (in the 1960's) and behavior relevant to QCD.

(12.4) Dimensionally: $$\boxed{l_s = \hbar c \sqrt{\alpha'}}$$