SUPERSTRING THEORIES (Z §13.5)

Last time, we finished a two-lecture exposition of basic ideas involved in quantizing (bosonic) strings. In particular, we met concepts of :-

- promoting $(X^I, \mathcal{P}^{\tau J})$ to operators, with CCRs

(1.1) $$[X^I(\sigma), \mathcal{P}^{\tau J}(\sigma')]_{\tau=\tau'} = i\eta^{IJ}\delta(\sigma-\sigma')$$

- seeing α^I_{-n} as $\sqrt{n}\times$(creation op. for n^{th} mode, $a_n^\dagger$)
- calculating $\perp$ Virasoro generators

(1.2) $$L_0^\perp = \alpha' p^I p^I + \sum_{p=1}^{\infty} p\,(a_p^I)^\dagger a_p^I$$

(1.3) with algebra $$[L_m^\perp, L_n^\perp] = (m-n)L^\perp_{m+n} + \frac{(D-2)}{12}(m^3-m)\,\delta_{m+n,0}$$

- Lorentz generators close ($[M^{-I}, M^{-J}] \to M$'s)

(1.4) $$\Rightarrow \frac{(D-2)}{24} = 1 \quad \text{and so} \quad H = L_0^\perp - 1$$

- Tachyon, gauge field A_μ at $m^2 \leq 0$ (open)
- Tachyon, graviton $g_{\mu\nu}$, 2-form $B_{[\mu\nu]}$, dilaton Φ (closed)

Today, in part I of lecture, I will give a brief exposition of what changes when we add fermionic degrees of freedom. It will not be possible to be a great deal more detailed, because of time constraints and because this is an introductory course, but I will develop some of the more powerful ideas in the last couple of lectures or so. ☺

So, what's the basic idea? SUSY says :-

- Partner every boson field, like $X^I(\tau,\sigma)$ with a fermion field of the same mass, like

(1.5) $$\boxed{\psi^I_\alpha(\tau,\sigma)\ , \quad \alpha = 1,2}$$

This field does not satisfy canonical commutation relations, but rather canonical anticommutation relations; ψ is a fermion field. Classically, ψ is an anticommuting field, e.g. two fermions b_1, b_2 obey $b_1 b_2 = -b_2 b_1$. For this reason, you can NOT build up big coherent states of fermion fields — physically this is encoded in the Pauli Exclusion Principle (PEP): no two fermions can occupy the same quantum state. This makes fermions completely unlike photons or gravitons, for example.

- Status of ψ^I_α under Lorentz transformations?

Very non-trivial issue: ψ^I_α has a vector spacetime index to start with, as well as spin-half properties in the QFT of the string worldsheet: i.e. it is a worldsheet spinor. How do we square that fact with developing a spacetime supersymmetry?

⊛ This is the "magic" of light-front quantization in which $\psi^+_\alpha = 0$ allows solving for ψ^-_α in terms of transverse ψ^I_α; along the way ψ^I_α actually get transmogrified by quantization into a spacetime spinor. (This is REALLY COOL!)

⊛ ψ^μ_α, because it is a spacetime vector, gets transformed by $M^{\mu\nu}$ and hence extra fields (ψ) get involved in cancelling anomalous terms in $[M^{-I}, M^{-J}]$. In addition, there are new fermionic generators Q which extend the Lorentz algebra in a fermionic direction.

The consistency conditions now read

(2.1) $\boxed{D=10}$ (superstring) (rather than $D=26$ for bosonic case: we had $a=-1$)

and

(2.2) $\boxed{a=-\frac{1}{2}}$

Equations of motion for ψ^I_α are first-order. (Massless) fermions obey wave equation ⇒

(3.1a) $\boxed{\psi_1^I \text{ is right-moving}}$ $\left[(\partial_\tau - \partial_\sigma)\psi_1^I = 0\right]$

(3.1b) and $\boxed{\psi_2^I \text{ is left-moving}}$ $\left[(\partial_\tau + \partial_\sigma)\psi_2^I = 0\right]$

This means that we have two on-shell degrees of freedom; these partner up to X_L, X_R in $X^I = X_R^I(\tau-\sigma) + X_L^I(\tau+\sigma)$ (oscillators, all included).

OPEN STRING

(3.2) $$\psi_1^I(\tau,\sigma) = \psi_1^I(\tau-\sigma), \quad \psi_2^I(\tau,\sigma) = \psi_2^I(\tau+\sigma)$$

Endpoints must satisfy requirement from action principle that

$$\psi_1^I(\tau,\sigma_*)\,\delta\psi_1^I(\tau,\sigma_*) - \psi_2^I(\tau,\sigma_*)\,\delta\psi_2^I(\tau,\sigma_*) = 0$$

at endpoints.

(3.3) Ansätz require $\boxed{\psi_1^I(\tau,\sigma_*) = \pm\, \psi_2^I(\tau,\sigma_*)}$ and then $\delta\psi$ also satisfy this relation.

(3.4a) Convention: $\psi_1^I(\tau,0) = + \psi_2^I(\tau,\pi)$ ← "RAMOND"

(3.4b) and ∴ $\psi_1^I(\tau,\pi) = - \psi_2^I(\tau,\pi)$ ← "NEVEU-SCHWARZ"

(Nothing to do with behaviour of spinor fields under rotations!)

Can simplify by assembling dynamical (open-string) field

(3.5) $$\boxed{\Psi^I_{\text{open}}(\tau,\sigma) = \begin{cases} \psi_1^I(\tau,\sigma), & \sigma \in [0,\pi] \\ \psi_2^I(\tau,-\sigma), & \sigma \in [-\pi,0] \end{cases}}$$

Ψ^I continuous at $\sigma=0$ by (3.1 a, b). (✓)

then NS/R condition says

(3.6) $$\boxed{\Psi^I(\tau,\pi) = \pm\, \Psi^I(\tau,-\pi) \Leftrightarrow \begin{cases} \text{R periodic} \\ \text{NS anti-periodic} \end{cases}}$$

Neveu-Schwarz sector (quantum theory)

$\Psi^I(\tau,\sigma)$ are HALF-integrally moded fields:

(4.1) $$\boxed{\Psi^I(\tau,\sigma) = \sum_{r\in \mathbb{Z}+\frac{1}{2}} b_r^I \, e^{-ir(\tau-\sigma)} \quad (NS)}$$

$b^I_{-1/2+n}$, $n \in \mathbb{Z}^+ \cup \{0\}$ are creation operators

and b^I_r anticommute so e.g. $(b^I_{-r})^2 = 0$

$\Rightarrow$ Oscillating modes with basis

(4.2) $$\boxed{\underline{NS}: \prod_{I=2}^{9} \prod_{n=1}^{\infty} (\alpha^I_{-n})^{\lambda_{n,I}} \prod_{J=2}^{9} \prod_{r=1/2,3/2,5/2,\ldots} (b^J_{-r})^{\rho_{r,J}} \, |NS\rangle \otimes |p^+, \vec{p}_T\rangle}$$

Ramond sector

(4.3) $$\boxed{\Psi^I(\tau,\sigma) = \sum_{n\in\mathbb{Z}} d^I_n \, e^{-in(\tau-\sigma)} \quad (R)}$$

Zero modes have to be treated ; d_0^I can be arranged (à la $d_0^1 \pm i d_0^2$; etc) into four complex creation/annihilation operators with algebra.

(4.4) $\Rightarrow 2^4 = 16$ degenerate R groundstates $\boxed{\{|R^A\rangle, \ A=1,\ldots,16\}}$

and

(4.5) $$\boxed{\underline{R}: \prod_{I=2}^{9} \prod_{n=1}^{\infty} (\alpha^I_{-n})^{\lambda_{n,I}} \prod_{J=2}^{9} \prod_{m=1}^{\infty} (d^J_{-m})^{\rho_{m,J}} \, |R^A\rangle \otimes |p^+, \vec{p}_T\rangle}$$

Open string state space

Take subset of R sector + subset of R sector (Consistent truncation!)

(4.6) $\boxed{\text{NS : odd fermion \#}}$

All states have same # fermion oscillators, mod 2.

GSO projection

(5.1) Ramond ?

$$\boxed{R: \left\{ (d_{-n}\text{'s on}) |R_1^a\rangle \text{ with } \underset{\sim}{\text{even}} \ \# \ d^I_{-n} \right\} \oplus \left\{ (d_{-m}\text{'s on}) |R_2^a\rangle \text{ with } \underset{\sim}{\text{odd}} \ \# \ d^I_{-m} \right\}}$$

Spacetime

GSO ⇒ { fermions from R sector / bosons from NS sector } open strings

$$\boxed{\begin{array}{l} \{A^\mu\} \leftrightarrow b^I_{-1/2} |NS\rangle \otimes |p^+, \vec{p}_T\rangle \\ \alpha' M^2 = -\frac{1}{2} + N^\perp \qquad (NS) \end{array}}$$

$$\alpha' M^2 = 0 \qquad (R)$$

$$\boxed{\{\lambda_\alpha\} \leftrightarrow |R_1^a\rangle \otimes |p^+, \vec{p}_T\rangle}$$
(8 cpts by GSO)

gaugino ; A^μ and λ_α have eight (physical) components on-shell

CLOSED STRING

(L) : $\left\{ \begin{array}{l} |NS\rangle_L \\ |R\rangle_L \end{array} \right\}$; (R) : $\left\{ \begin{array}{l} |NS\rangle_R \\ |R\rangle_R \end{array} \right\}$.

⇒ four types of closed superstring states.

GSO projection : odd # fermionic oscillators on $|NS\rangle$ and either $\left\{ \begin{array}{l} \text{even } \# \text{ on } |R_1^a\rangle \\ \text{or odd } \# \text{ on } |R_2^a\rangle \end{array} \right.$
This gives spacetime susy.
(other projections don't!)

$$\boxed{g_{\mu\nu}, B_{(2)}, \Phi \text{ from } b^I_{-1/2} \bar{b}^J_{-1/2} |NS\rangle_L \otimes |NS\rangle_R \otimes (p^+, \vec{p}_T)}$$

(6.1)
$$\boxed{\begin{array}{l} \text{R-NS} + \text{NS-R} \iff \text{spacetime fermions} \\ b^I_{-1/2}|NS\rangle_L \otimes |R^b_j\rangle_R \otimes (p^+, \vec{p}_T) \\ b^I_{-1/2}|R^a_i\rangle_L \otimes |NS\rangle_R \otimes (p^+, \vec{p}_T) \end{array}}$$

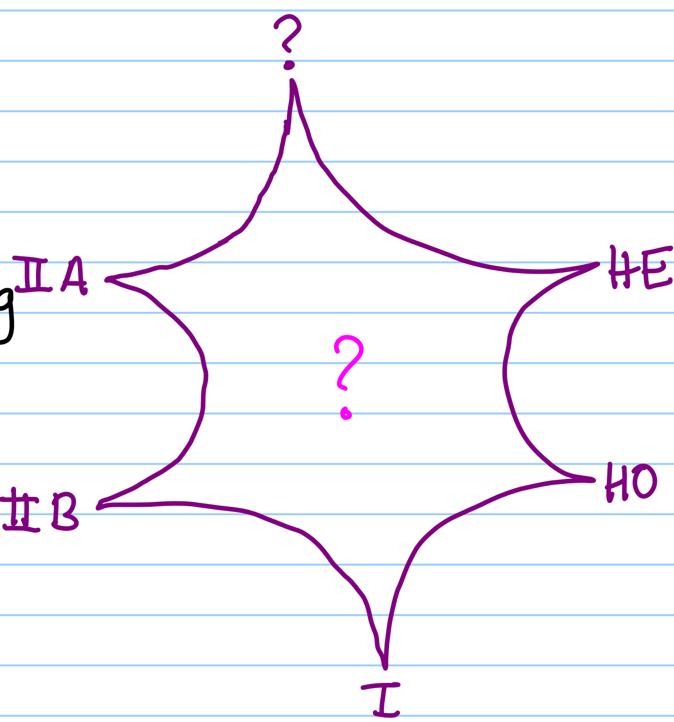
128 of them: $2\times(8\times 8)$ $\quad i = 1..8$

while

(6.2)
$$\boxed{\begin{array}{llll} \text{R-R} \iff \text{extra bosonic fields} & & & \\ & A_{[n]}, & n \text{ odd} & \text{IIA (non-chiral)} \\ & A_{[m]}, & m \text{ even} & \text{IIB (chiral)} \end{array}}$$

Other theories with SUSY

closed superstring



closed superstring

open superstring

heterotic (half superstring, half boson; extra fields → gauge bosons: consistency (anomaly cancellation) ⇒ rank = 496. $E_8\times E_8$ or $SO(32)$ $d_{\text{spacetime}}$ still = 10 (both L & R dudes available).

D-BRANES & GAUGE FIELDS (Z §14)

Dirichlet BC on endpoints rather than N, for (D−p) coords, led to hypersurfaces of p dimensions, called Dp-branes. spacetime on them has d=p+1.

(7.1) $\Rightarrow$
$$\boxed{\begin{array}{lll} X^a(\tau,0) = \bar{x}^a = X^a(\tau,\pi) & \text{"DD"}, & a=p+1,\dots,D-1 \\ X^{m\prime}(\tau,\sigma) = 0 = X^{m\prime}(\tau,\pi) & \text{"NN"}, & m=0,\dots,p \end{array}}$$

⊛ For quantization, everything we previously derived goes through very simply for NN. In particular, X^- is a NN coordinate.

(7.2) Then
$$\boxed{\dot{X}^i \pm X^{i\prime} = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \alpha_n^i\, e^{-in(\tau\pm\sigma)} \qquad (NN)}$$

Zwiebach simply explains what differs for DD coords on p. 278. The outcome of his math is

(7.3)
$$\boxed{(DD) \qquad X^a(\tau,\sigma) = \bar{x}^a + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^a e^{-in\tau}\sin(n\sigma)}$$

no zero mode here.

Contrast this to NN coords which have term in (CM momentum) × τ, encoding CM motion. Dirichlet BCs tie down our D-branes' CM. to be $\bar{x}^a$, $\partial_\tau \bar{x}^a = 0$.

(7.4) $\Rightarrow$
$$\boxed{X^{a\prime} \pm \dot{X}^a = \sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^a\, e^{-in(\tau\pm\sigma)} \qquad (DD)}$$

note relative ± sign to NN case, & n≠0.

DD mode quantization

(7.5)
$$\boxed{\left[X^a(\tau,\sigma), \frac{\dot{X}^b}{2\pi\alpha'}(\tau,\sigma')\right] = i\,\delta^{ab}\,\delta(\sigma-\sigma') \qquad (DD)}$$

(8.1) and $$\boxed{[\alpha^a_m, \alpha^b_n] = m\delta^{ab}\,\delta_{m+n,0}}$$

(8.2) then $$\boxed{2p^+p^- = \frac{1}{\alpha'}\left\{\alpha' p^i p^i + \sum_{n=1}^{\infty}\left[\alpha^i_{-n}\alpha^i_n + \alpha^a_{-n}\alpha^a_n\right] - 1\right\}}$$

(8.3) * Since only the 0 mode structure changed, same critical dimension.

and then

(8.4) $$\boxed{\alpha' M^2 = -1 + \sum_{n=1}^{\infty}\sum_{i=2}^{p} n(a^i_n)^\dagger a^i_n + \sum_{m=1}^{\infty}\sum_{a=p+1}^{D-1} m(a^a_m)^\dagger a^a_m}$$

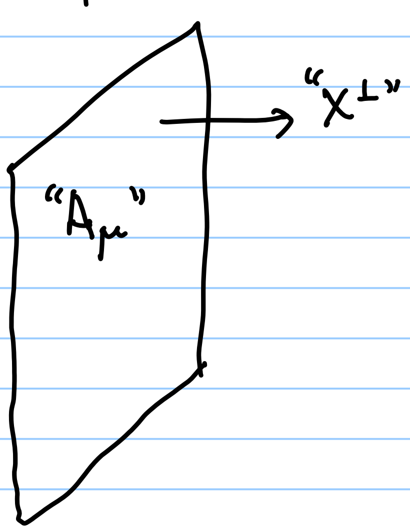
(8.5) $\Rightarrow$ states $$\boxed{\prod_{n=1}^{\infty}\prod_{i=2}^{p}\left((a^i_n)^\dagger\right)^{\lambda_{n,i}} \prod_{m=1}^{\infty}\prod_{a=p+1}^{D-1}\left((a^a_m)^\dagger\right)^{\lambda_{m,a}} |p^+, \vec{p}_T\rangle}$$

Since there is still [a tachyon and] a massless vector field, we conclude

(8.6) A Dp-brane has a Maxwell field living on its worldvol. + a massless scalar for each $\perp$ dir^n.

and

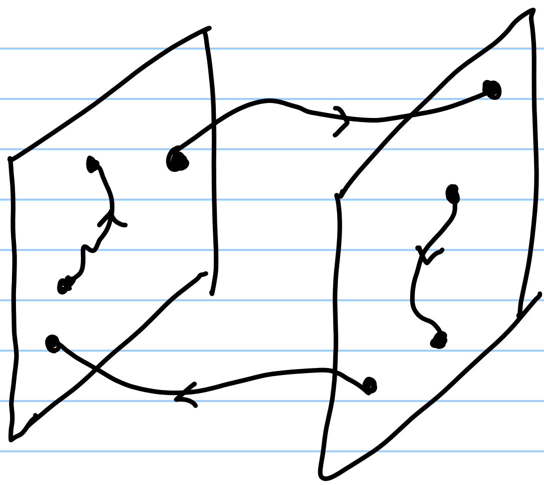
These correspond to motions of the D-brane in $\perp$ directions.



A_μ tell us about least expensive 'wiggles' on the worldvolume.

Note: D25 can't be moved ⟺ has no $X^\perp$ ☺

More than one D-brane.



For this case there are four sectors of fields.
(when open string worldsheet supports orientability.)

What changes?

DD coords have $X^a(\tau, 0) = \bar{x}_1^a$ but $X^a(\tau,\pi) = \bar{x}_2^a \neq \bar{x}_1^a$.

So compute mode expansion & m^2 using same algorithm as before, but this time

$$\boxed{\alpha' M^2 = (N^\perp - 1) + \left(\frac{\bar{x}_2^a - \bar{x}_1^a}{2\pi\alpha'}\right)^2}$$

But recall: tension of F1 is $\frac{1}{2\pi\alpha'}$.

So extra piece in M^2 is just (length of stretched string) × (tension) ☺.

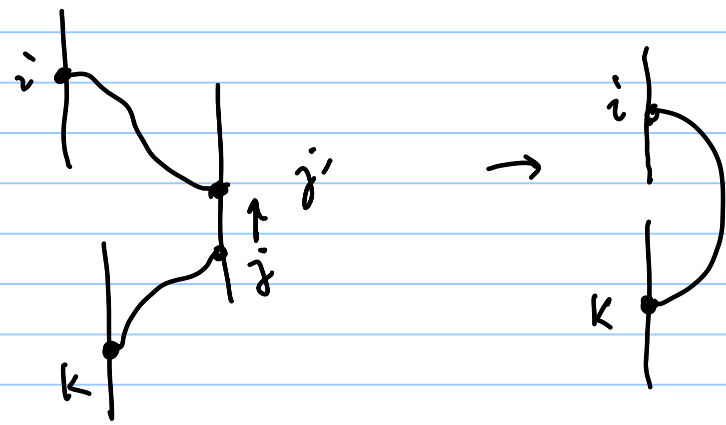
Construct ground states distinguished by label $[ij]$ according to which brane start & end upon.

Then have $|p^+, \vec{p}_T ; [ij]\rangle$ $\quad i,j \in \{1,2\}$

Oscillator structure exactly same $\forall i,j$

We will see later in more detail that Chan-Paton factors carrying $\underline{n}$ of $U(n)$ for n D-branes can be tacked onto open-string endpts without costing energy.

n^2 independent fields.



so $[ij] * [jk] = [ik]$

$\Leftrightarrow$ well-defined (matrix) multiplication

The really amazing thing is that, by calculating further, we can discover that

Higgsing the $U(N)$ $\leftrightarrow$ separating D-branes!

Next time: we'll talk about

Dp-branes and Dq-branes together & $\parallel$;

$Dp \perp Dq$ (n), and so forth. ☺