Quantizing the relativistic particle (prelude to string)

Our focus will be on
(1) requiring Heisenberg operators in quantum theory to satisfy classical equations of motion;
(2) showing that the Schrödinger equation for the particle wavefunctions obey the classical field eqn;
(3) finding the quantum states and matching up with the scalar field one-particle states;
(4) setting up the generators of Lorentz symmetries in the light-cone gauge.

Let's do these one by one! ☺

Note on units: work with traditional particle physics units in which $\hbar = c = 1$
($\Leftrightarrow$ we're fully in the quantum relativistic régime)

Start with our friend the geometric action

(1.1) $$S_p = -m \int d\tau \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}$$

chosen to be dimensionless

the thingy under the square root is just the square of the 4-velocity $\dot{x}^\mu$ $(\cdot \equiv d/d\tau)$, because $\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \dot{x}^2$

i.e.,

(1.2) $$S_p = -m \int d\tau \sqrt{-\dot{x}^2}$$

(1.3) $$= \int d\tau\, L_p \quad \text{where} \quad L_p = -m\sqrt{-\dot{x}^2}$$

Canonical momenta are

(1.4) $$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = -\frac{1}{2} m \left(-\dot{x}^2\right)^{-1/2} \cdot -2\dot{x}_\mu = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}} \Rightarrow \boxed{p_\mu = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}}}$$

The Euler-Lagrange equations are

(1.5) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = \frac{\partial L}{\partial x^\mu} \Rightarrow p_\mu = \text{constant} : \frac{dp_\mu}{d\tau} = 0$$

all of them are conserved in τ.

Light-cone gauge choice

for the units' sake

(2.1) Sets $\boxed{x^+ = \frac{1}{m^2} p^+ \tau}$

Consider equation (1.4) and let's "pick on" its + components

(2.2) $p_+ = \frac{m\dot{x}_+}{\sqrt{-\dot{x}^2}}$ } let's combine

(2.3) but $x^+ = \frac{p^+\tau}{m^2}$

So track $p^+ = \frac{m\dot{x}_+}{\sqrt{-\dot{x}^2}} = \frac{1}{\sqrt{\dot{x}^2}}\frac{p^+}{m}$

$\Rightarrow m^{-2} = \dot{x}^2$ (in this set of units)

(2.4) or simply $\boxed{p^2 + m^2 = 0}$

Momentum is just

(2.5) $p_\mu = m^2 \dot{x}_\mu$

Equation of motion for x^μ and p_μ are:-

(2.6) $\ddot{x}^\mu = 0$

Equation (2.4) in light-cones mode gives

(2.7) $-2p^+p^- + p^I p^I + m^2 = 0 \Rightarrow \boxed{p^- = \frac{1}{2p^+}(p^I p^I + m^2)}$

whereupon (using (2.5))

(a) $\frac{dx^-}{d\tau} = \frac{1}{m^2} p^-$

(2.8) $\Rightarrow \boxed{x^-(\tau) = x_0^- + \frac{p^-}{m^2}\tau}$

(2.9) (b) $\frac{dx^I}{d\tau} = \frac{p^I}{m^2} \Rightarrow \boxed{x^I(t) = x_0^I + \frac{p^I}{m}\tau}$

All up: dynamical variables: (x^I, x_0^-, p^I, p^+)

↙ Intro for mathies; reminder for phyzzies ☺

Heisenberg & Schrödinger pictures

Time-evolution in QM can be handled in two equivalent ways –

(1) let the operators evolve & states stay fixed in t (Heisenberg)

(2) let the states evolve & operators stay fixed* (Schrödinger)

Heisenberg picture is closer to the spirit of the

(3.1) classical mechanics. The Poisson bracket $\{q,p\}_{P.B.} = 1$

(3.2) goes to $[q,p] = i\hbar$

(3.3) and in Heisenberg picture this is $\boxed{[q(t),p(t)] = i\hbar}$

Then the Hamilton equation of motion for some Heisenberg operator $\mathcal{O}_H$ is

(3.4) $$i\frac{d}{dt}\mathcal{O}_H = [\mathcal{O}_H(t), H(q(t),p(t);t)]$$

In Schrödinger picture the states do the evolving:

(3.5) $$|\Psi,t\rangle = e^{-iHt}|\Psi\rangle$$

(3.6) satisfies $$i\frac{d}{dt}|\Psi,t\rangle = H|\Psi,t\rangle.$$

This is called the Schrödinger equation.

Schrödinger operators $\mathcal{O}_S$ get converted to Heisenberg ones

(3.7) via $$\mathcal{O}_H = e^{iHt}\mathcal{O}_S e^{-iHt}.$$

Note that $\mathcal{O}_H$ has no explicit time-dependence if it commutes with the Hamiltonian.

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* assuming that $\partial/\partial t\, H(p,q,t) = 0$ (no external forcing) i.e. we didn't forget to include relevant degrees of freedom in our action...

	Schrödinger ops	Heisenberg ops
(4.1) (a)	(x^I, x_0^-, p^I, p^+)	$(x^I(\tau), x_0^-(\tau), p^I(\tau), p^+(\tau))$
(b)	$[x^I, p^J] = i\eta^{IJ}$	$[x^I(\tau), p^J(\tau)] = i\eta^{IJ}$
(c)	$[x_0^-, p^+] = i\eta^{-+} = -i$	$[x_0^-(\tau), p^+(\tau)] = -i$ conjugate momentum, spatial coord
(d)	(all other CCR's = 0)	← (ditto)

- Other operators $x^+(\tau)$ $x^-(\tau)$ $p^-(\tau)$ are obtained via

(4.2) $$x^+(\tau) \equiv \frac{p^+\tau}{m^2} \;;\; x^-(\tau) \equiv x_0^- + \frac{p^-\tau}{m^2} \;;\; p^-(\tau) = \frac{1}{2p^+}(p^I p^I + m^2).$$

Time evolution operator?

Know $\partial_+ \leftrightarrow p^-$

but we're after ∂_τ here.

$$\frac{\partial}{\partial\tau} = \frac{p^+}{m^2}\frac{\partial}{\partial x^+} = \frac{p^+}{m^2}p^-$$

⇒ postulate that

(drumroll...! ☺)

(4.3) $$\boxed{H(\tau) = \frac{p^+(\tau)}{m^2}p^-(\tau) = \frac{1}{2m^2}\left(p^I(\tau)p^I(\tau) + m^2\right)}$$ VIP eqn!

From this and (4.1) (a)–(d), simple to check that

$$i\frac{d}{d\tau}p^+(\tau) = [p^+(\tau), H] = 0$$

$$i\frac{d}{d\tau}p^I(\tau) = [p^I(\tau), H] = 0$$

(using $[p^\mu, p^\nu] = 0$ $\forall p^\mu, p^\nu$)

property of translation group.

and

$$\frac{dx^I(\tau)}{d\tau} = \frac{p^I}{m^2} \quad \text{and so} \quad x^I(\tau) = x_0^I + \frac{p^I\tau}{m^2}$$

while

$$\frac{dx_0^-}{d\tau} = 0 \quad \text{i.e. } x_0^-(\tau) \text{ is a constant of motion}$$

Construct 1-particle states of quantum point particle labelled by momenta:

(5.1) $$|p^+, \vec{p}_T\rangle$$

- Eigenvalues of operators? e.g.: $\hat{p}^+|p^+,\vec{p}_T\rangle = p^+|p^+,\vec{p}_T\rangle$.

- Hamiltonian acts as
$$\hat{H}\,|p^+,\vec{p}_T\rangle = \frac{1}{2m^2}(\vec{p}_T\cdot\vec{p}_T + m^2)\,|p^+,\vec{p}_T\rangle$$

$\Rightarrow$ • states of the form

(5.2) $$|\Psi(\tau)\rangle \equiv \exp\left(-\frac{i}{2m^2}(\vec{p}_T\cdot\vec{p}_T + m^2)\tau\right)|p^+,\vec{p}_T\rangle$$

satisfy the Schrödinger equation

- We can always build momentum-space wavefunctions

(5.2a) $\Rightarrow$ write $|\Psi,\tau\rangle = \int dp^+\, d\vec{p}_T\, \psi(\tau,p^+,\vec{p}_T)\,|p^+,\vec{p}_T\rangle$.

Defining bras $\langle p^+,\vec{p}_T|$ dual to kets $|p^+,\vec{p}_T\rangle$ satisfying orthonormality conditions
$$\langle k^+,\vec{k}_T|p^+,\vec{p}_T\rangle = \delta(k^+-p^+)\,\delta(\vec{k}_T-\vec{p}_T)$$

i.e. $\psi(\tau,p^+,\vec{p}_T) = \langle p^+,\vec{p}_T|\psi,\tau\rangle$

(5.3) $|\Psi,\tau\rangle$ satisfies $\boxed{i\frac{\partial}{\partial t}|\Psi,\tau\rangle = H\,|\Psi,\tau\rangle}$ Schrödinger equation

Matching up particle with field states

It's a deep and straightforward result that

(5.4) $$\boxed{|p^+,\vec{p}_T\rangle \leftrightarrow a^\dagger_{p^+,\vec{p}_T}|\Omega\rangle}$$

This is of course limited to 1-particle states of the scalar field we quantized before. ☺

So one-particle relativistic quantum physics is described as single-quantum of scalar field.

["I sense a disturbance in the Force, Luke...!"
"Obiwan, it's just a graviton passin' thru!"]

The scalar field in light-front gauge obeys, as we saw last time, a first-order equation:-

(6.1) $$\left[i\frac{\partial}{\partial \tau} - \frac{1}{2m^2}(p^I p^I + m^2) \right] \phi(\tau, p^+, \vec{p}_T) = 0.$$

This $\phi(\tau, p^+, \vec{p}_T)$ is precisely the Schrödinger wave-function $\psi(\tau, p^+, \vec{p}_T)$ in eqn (5.2a) above.

▷ This was all <u>first quantization</u>
Second quantization is when we permit creation and destruction of excitations of the quantum field; we can construct multi-particle states.

<u>Poincaré generators</u>

- Conserved charges ⇒ quantum #s.
- $\boxed{\text{Translations}}$ $\delta x^\mu = \epsilon^\mu$ ⇒ momentum p^μ conserved

Write this in more sophisticated (easily generalizable) way:

(6.2) $$\begin{cases} \delta x^\mu(\tau) = [i\epsilon_\rho p^\rho(\tau), x^\mu(\tau)] \\ \quad = i\epsilon_\rho(-i\eta^{\mu\rho}) = \epsilon^\mu \end{cases}$$

The action S_p was invariant (up to a total derivative) under reparametrizations

(6.3) $$\begin{cases} x^\mu(\tau) \to x^\mu(\tau + \lambda(\tau)) = x^\mu(\tau) + \lambda(\tau)\partial_\tau x^\mu(\tau) \\ \text{i.e.} \\ \delta x^\mu(\tau) = \lambda(\tau)\partial_\tau x^\mu(\tau) \end{cases}$$

So what do generators (p^+, p^-, p^I) <u>do</u>?

- p^- generates a translation and reparametrization. Safely, it does not alter x^+ which we set in light-cone gauge to be $\frac{p^+\tau}{m^2} = x^+$
- p^I generates translations on x^I only.

Light-cone Lorentz generators.

These come from
$$\delta x^\mu = \epsilon^{\mu\nu} x_\nu \qquad \text{where} \qquad \epsilon^{\mu\nu} = -\epsilon^{\nu\mu}.$$
Associated conserved* quantities are

(7.1) $$\boxed{M^{\mu\nu} = (x^\mu p^\nu - x^\nu p^\mu)(\tau)}$$ (from Noether's theorem).

* conserved classically! Later we do quantum story for string & compute the critical dimension.

(7.2) $$\boxed{(M^{\mu\nu})^\dagger = M^{\mu\nu}}$$

Then $$\boxed{\delta x^\rho = \left[-\frac{i}{2}\epsilon_{\mu\nu} M^{\mu\nu}, x^\rho\right]} = \epsilon^{\rho\nu} x_\nu$$

These and the p's satisfy the Poincaré algebra

(7.3) (a) $$[p^\mu, p^\nu] = 0, \quad [M^{\mu\nu}, p^\lambda] = i p^\mu \eta^{\nu\lambda} - i p^\nu \eta^{\mu\lambda},$$
(b) (c) $$[M^{\mu\nu}, M^{\lambda\sigma}] = i\eta^{\mu\lambda} M^{\nu\sigma} - i\eta^{\nu\lambda} M^{\mu\sigma} + i\eta^{\mu\sigma} M^{\lambda\nu} - i\eta^{\nu\sigma} M^{\lambda\mu}$$

- Light-cone has simplifications from metric structure. Also get subtleties, with defining M^{+-} and M^{-I}; problem is with hermiticity. To solve this problem, best definition is (solving (7.3)!)

(7.4) (a) $$\boxed{M^{+-} = -\frac{1}{2}(x_0^- p^+ + p^+ x_0^-)}$$
(b) and $$\boxed{M^{-I} = x_0^- p^I - \frac{1}{2}(x_0^I p^- + p^- x_0^I)}$$ (operator ordering ambiguity)

woo hoo !!

OPEN STRING QUANTIZATION (RELATIVISTIC)

We found clever parametrizations for worldsheet coords τ and σ, and saw light front gauge is straight-forward.

(8.1) Had: $\mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'} X'^{\mu}$; $\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \dot{X}^{\mu}$;

(8.2) $X^+ = 2\alpha' p^+ \tau$.

Using constraint equation from last time we have

(8.3) $$\dot{X}^- = \frac{1}{2\alpha'}\frac{1}{2p^+}\left(\dot{X}^I\dot{X}^I + X'^I X'^I\right)$$

Let's work in Heisenberg picture (Z. does Schrö too.)

(8.4) Operators are: $\left(X^I(\tau,\sigma),\ x_0^-(\tau),\ \mathcal{P}^{\tau I}(\tau,\sigma),\ p^+(\tau)\right)$

Nonvanishing equal-time CCR's are

(8.5) $$\boxed{\left[X^I(\tau,\sigma), \mathcal{P}^{\tau J}(\tau,\sigma')\right] = i\eta^{IJ}\delta(\sigma-\sigma')}$$

τ ensures physics "local" along string

Hamiltonian is conserved:

$$H = 2\alpha' p^+ p^- = 2\alpha' p^+ \int_0^{\pi} d\sigma\, \mathcal{P}^{\tau -}$$

Using (8.3), H 'simplifies' :-

(8.6) $$\boxed{H(\tau) = \pi\alpha' \int_0^{\pi} d\sigma \left\{ \mathcal{P}^{\tau I}(\tau,\sigma)\mathcal{P}^{\tau I}(\tau,\sigma) + \frac{X^{I\prime}(\tau,\sigma)X^{I\prime}(\tau,\sigma)}{(2\pi\alpha')^2}\right\} = L_0^{\perp}}$$

So the Hamiltonian, the generator of time translations on the worldsheet, is the zeroth Virasoro operator; in the transverse directions.

We can check whether (8.6) gives expected results.

In general, the right H generates time evolution:

$$i\dot{\xi}(\tau,\sigma) = [\xi(\tau,\sigma), H] \quad \text{for arbitrary } \xi(\tau,\sigma)$$

This actually shows that $\dot{H}=0 \because [H,H]=0$.

(9.1) $\Rightarrow$ $\boxed{H = \text{const.}}$

(9.2) Also, assuming that $\boxed{[x_0^-, p^+] = -i}$ $(\hbar = 1)$

get

(9.3) $\dot{X}^I = 2\pi\alpha' \mathcal{P}^{\tau I}$ (sim. for $\dot{\mathcal{P}}^{\tau I}$ ...)

$\Rightarrow$

(9.4) $\ddot{X}^I - X^{I''} = 0$ with either D or N BC's.

These last 2 equations should be mega-familiar.

- Eqns (8.5) & (9.2) show us the commutation relations for the X's. Let's figure out the CCR's of oscillators.

$\downarrow$

Expand out open-string position fields

(9.5) $$X^I(\tau,\sigma) = x_0^I + \sqrt{2\alpha'}\,\alpha_0^I \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^I \cos(n\sigma) e^{-in\tau}$$

so that

(9.6) $$(\dot{X}^I \pm X'^I) = \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \alpha_n^I e^{-in(\tau\pm\sigma)}$$

(Notice that the RHS here naturally defines a periodic function with periodicity 2π over σ.
$\Rightarrow$ use a neat trick to "make it so" by writing

(9.7) e.g. $$\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \alpha_n^I e^{-in(\tau+\sigma)} = \begin{cases} (\dot{X}^I + X^{I'})(\tau,\sigma) & \sigma\in[0,\pi] \\ (\dot{X}^I - X^{I'})(\tau,-\sigma) & \sigma\in[-\pi,0] \end{cases}$$)

Algebra $\Rightarrow$

$$[(\dot{X}^I \pm X^{I'})(\tau,\sigma), (\dot{X}^J \pm X^{J'})(\tau,\sigma')] = \pm 4\pi\alpha' i \eta^{IJ} \frac{d}{d\sigma}\delta(\sigma-\sigma')$$

$(\sigma \in [0,\pi])$

while $[(\dot{X}^I \pm X^{I\prime})(\tau,\sigma), (\dot{X}^J \mp X^{J\prime})(\tau,\sigma')] = 0$

Then (more algebra)

(10.1) $$\boxed{[\alpha_m^I, \alpha_n^J] = m\eta^{IJ}\delta_{m+n,0}}$$

(10.2) $\boxed{\alpha_0^I = \sqrt{2\alpha'}\,p^I}$ commutes w̄ all.

(10.3) and $\boxed{[x_0^I, p^J] = i\eta^{IJ}}$

Oscillator mode operators also obey

(10.4) $$\boxed{(\alpha_n^I)^\dagger = \alpha_{-n}^I \quad , \; n \in \mathbb{Z}}$$

α_n^I are	destruction	operators	
α_{-n}^I are	creation	operators	$(n \geq 1)$

so that

$$X^I(\tau,\sigma) = x_0^I + 2\alpha' p^I \tau + i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left(\frac{1}{n}\alpha_n^I e^{-in\tau} - \frac{1}{n}\alpha_{-n}^I e^{in\tau}\right)\cos(n\sigma)$$

In other words, quantization of strings involves a whole (infinite) collection of simple harmonic oscillators; a SHO being something near and dear to every physicist's heart ♡ ☺

Read Zwiebach §12.3 to get a step-by-step derivation of this if you don't find my handware convincing ☺

Virasoro algebra (first look)

In LF gauge we set $X^+ \propto \tau$ and used constraints to get X^- and hence α_n^- operators:

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'}\,p^+} L_n^\perp$$

where

(11.1) $$L_n^\perp \equiv \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha^I_{n-p}\,\alpha^I_p$$ ← true as a quantum operator expression now, except operator-ordering ambiguity subtle.

To make the ambiguity alive, let's focus on $L_0^\perp$, which generates worldsheet ∂_τ. We have

$$L_0^\perp = \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha^I_{-p}\,\alpha^I_p$$

(11.2) $$= \frac{1}{2}\alpha_0^I\,\alpha_0^I + \frac{1}{2}\sum_{p\in\mathbb{N}} \alpha^I_{-p}\,\alpha^I_p + \frac{1}{2}\sum_{p\in\mathbb{N}} \alpha^I_p\,\alpha^I_{-p}$$

But we also know the CCR's

(11.3) $$[\alpha^I_m, \alpha^J_n] = m\,\eta^{IJ}\,\delta_{m+n,0}$$ so that

(11.4) $$L_0^\perp = \frac{1}{2}\alpha_0^I\,\alpha_0^I + \sum_{p\in\mathbb{N}} \alpha^I_{-p}\,\alpha^I_p + \underbrace{\sum_{p\in\mathbb{N}} p\cdot\frac{1}{2}(D-2)}_{\text{Call this ``}-a\text{''}}$$ ← #, X^I fields

$$\therefore\ (L_0^\perp + a) = \alpha' p^I p^I + \sum_{p\in\mathbb{N}} \alpha^I_{-p}\,\alpha^I_p$$

$$= \alpha' p^I p^I + \sum_{p\in\mathbb{N}} p\,\underbrace{a^I_{-p}\,a^I_p}_{\#\text{ operator}} \qquad a^I_p \equiv \frac{\alpha^I_p}{\sqrt{n}}$$

Then

$$m^2 = -p^2 = 2p^+p^- - p^Ip^I = (L_0^\perp + a)/\alpha' - p^Ip^I$$

$$\Rightarrow \boxed{\alpha' M^2 = \sum_{n=1}^{\infty} n\, a^I_{-n}\,a^I_n + a}$$

How will we deal with a, that encodes ZPE's for all oscillators in the X^I fields?! It's ∞!

(12)

Riemann ζ-function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ $\mathrm{Re}(s) > 1$.

(relevant for our zero-point energy sum)

$\zeta(-1) = -\frac{1}{12}$ (!)

This implies that $\boxed{a = \frac{1}{2}(D-2)\left(-\frac{1}{12}\right) = -\frac{(D-2)}{24}}$

We will later derive this using a more rigorous (and more believable!) method when we've constructed the Lorentz generators.

N.B. ① $\alpha' M^2 = \sum_{n \in \mathbb{N}} n \, (a_n^I)^\dagger a_n^I - a$

$\Rightarrow$ $\alpha' M^2 = -1$ $\Leftrightarrow$ no oscillators. Tachyon!

We will soon give real physical interpretation.

② If there is one unit of oscillation in the string state, say built via $(a_1^I)^\dagger |\Omega; k^+, \vec{k}_T\rangle$ then

$\alpha' M^2 = 0$ $\Leftrightarrow$ one unit of oscillation.

This oscillation has to point in some direction and so it has a vector index.

It is the massless gauge field whose quanta are photons!

Last comment for now: $(L_m^\perp)^\dagger = (L_{-m}^\perp)$

Next time we'll work out the algebra of the $L_m^\perp$ = the Virasoro algebra.

We'll also find the Lorentz generators, and construct state space generally.

Then we'll go & do it all again for closed strings!