Light-Front Coordinates

For purposes of quantizing our string (later on), we will need to get familiar with a particular choice of (τ,σ) different from the static gauge studied so far.

Here is a lightning  review of these concepts.

(1.1) Define $x^\pm \equiv \frac{1}{\sqrt{2}}(x^0 \pm x^1)$

Then $-(dx^0)^2 + (dx^1)^2 = -2\,dx^+dx^-$

(1.2) so that the spacetime metric is $\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (\eta_{\mu\nu})$

Light-cone components of any vector or tensor can easily be found. Also,

(1.3) $$a\cdot b = -a^+b^- - a^-b^+ + a^2b^2 + a^3b^3$$

• In light-cone coords, let us inspect what physicists call the mass shell condition (it's one of two invariants of Poincaré: $p^\mu p_\mu$.)

$$p^\mu p_\mu = \eta_{\mu\nu}p^\mu p^\nu = -p^-p^+ - p^+p^- + \vec{p}_\perp^{\,2}$$

$$= -2p^-p^+ + \vec{p}_\perp^{\,2} = -m^2$$

$$\Rightarrow \quad p^-p^+ = \frac{1}{2}(m^2 + \vec{p}_\perp^{\,2})$$

(1.4) $$\boxed{-p_+ = p^- = \frac{1}{2p^+}(m^2 + \vec{p}_\perp^{\,2})}$$ "looks" non-relativistic (!)

(1.5) (where $p^\pm = \frac{1}{\sqrt{2}}\left(\frac{E}{c}, \pm p^1, 0, 0\right)$)

• Light-cone time is taken to be x^+.
$\Rightarrow$ p_+ is the light-cone energy

String gauge choices: noncovariant gauges

Linear combination of position fields set to $\lambda\tau$:-

(2.1) $$\boxed{n_\mu X^\mu(\tau,\sigma) = \lambda\tau}$$

Breaks Lorentz invariance (via choice of particular n_μ).

The set of points x^μ satisfying $n_\mu x^\mu = \lambda\tau$ forms a hyperplane normal to n_μ.
$\Rightarrow$ the curve where the string worldsheet intersects this hyperplane normal to n_μ is the string at fixed τ.

What does this imply about energy-momentum? Static gauge! The momentum density was $\mathcal{P}^{\tau\mu}$, where energy is taken as "∂/∂(time)". If instead we use the gauge (2.1), what we should demand is conservation not of $\mathcal{P}^{\tau\mu}$ but

(2.2) $$n_\mu \mathcal{P}^{\tau\mu} = \text{constant in } \tau \text{ \& } \sigma$$

This makes the string have constant energy density along its length (tension is constant in σ parametrization); the eqn (2.2) also ensures τ and σ are nicely orthogonal as we demanded earlier. Zwiebach shows that for an open string we can take $\sigma \in [0,\pi]$.

Equation of motion?

$$\frac{\partial}{\partial\tau}\mathcal{P}^{\tau\mu} + \frac{\partial}{\partial\sigma}\mathcal{P}^{\sigma\mu} = 0$$

$$\Rightarrow \frac{\partial}{\partial\tau}(n\cdot\mathcal{P}^\tau) = -\frac{\partial}{\partial\sigma}(n\cdot\mathcal{P}^\sigma)$$

(2.3) $$= 0 \Rightarrow (n\cdot\mathcal{P}^\sigma) \text{ independent of } \sigma.$$

For an open string, we required for free endpoints $\mathcal{P}^{\sigma\mu} = 0$ at the endpoints. But this implies $(n\cdot\mathcal{P}^\sigma) = 0$ at endpoints. Independence of σ then gives

co) $$n\cdot\mathcal{P}^\sigma = 0 \text{ everywhere for open string (!)}$$

But for closed strings we have different BC's. Had

Momentum (Conserved): $p_\mu = \int_\gamma (\mathcal{P}^\tau_\mu d\sigma - \mathcal{P}^\sigma_\mu d\tau)$

where γ winds once around cylindrical-topology worldsheet.

Now, choose purely spatial path along $\partial/\partial\sigma$:

(3.1) Let $\boxed{(n\cdot p)\,\sigma = \frac{2\pi}{\beta}\int_0^\sigma d\tilde{\sigma}\; n\cdot\mathcal{P}^\tau(\tau,\tilde{\sigma})}$

← to make $\begin{cases}\sigma\in[0,2\pi] & \text{for closed strings}\\ \sigma\in[0,\pi] & \text{for open strings}\end{cases}$

while

(3.2) $\boxed{n\cdot X(\tau,\sigma) = \beta\alpha'(n\cdot p)\,\tau}$ ← gauge choice with right dimensions, and tension, etc.

For the closed string, is $n\cdot\mathcal{P}^\sigma = 0$ like for open string?

$$n\cdot\mathcal{P}^\sigma = n_\mu\left\{-\frac{1}{2\pi\alpha'}\left[\frac{(\dot{X}\cdot X')\partial_\tau X^\mu - \dot{X}^2\partial_\sigma X^\mu}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2(X')^2}}\right]\right\}$$

$$= -\frac{1}{2\pi\alpha'}\,\frac{(\dot{X}\cdot X')\,\partial_\tau(n\cdot X)}{\sqrt{(\dot{X}\cdot X')^2-\dot{X}^2(X')^2}} \quad \text{using (3.2)}$$

In order to make this vanish, we would need

(3.3) $$\dot{X}\cdot X' = 0$$

on at least one point along the string; that would be enough to make it zero everywhere along σ by (2.3). Note that (3.3) is a relativistic dot product.

Is it zero somewhere?

Let $\{t^\mu\}$ the tangent vector to the worldsheet at a point P then $\{x'^\mu\}$ and $\{t^\mu\}$ generate T_P; we know they are not parallel because X'^μ is spacelike while t^μ is timelike. If they are orthogonal, i.e. $t^\mu X'_\mu = 0$, then t^μ is the tangent vector doing the job. Otherwise, define

$$v^\mu = t^\mu - \left(\frac{t\cdot X'}{X'\cdot X'}\right)X'^\mu \quad \text{which is } \perp X'_\mu:\ v^\mu X'_\mu = 0.$$

Defining $\sigma=0$ to be the line given by

$$X^\mu(P) + \epsilon v^\mu$$

σ can be any point on string.

ensures that the tangent $\perp X'_\mu$ and, since the tangent vector $\propto \dot{X}^\mu$, have (3.3) satisfied. ⇒

$$\Rightarrow \boxed{n\cdot\mathcal{P}^\sigma = 0} \quad \text{(open \& closed)}$$

$\frac{\partial}{\partial\sigma}$ symmetry unfixed!

(4.1) Having $\boxed{\dot{X}\cdot X' = 0}$ simplifies life.

$$\Rightarrow \ \mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \frac{(X')^2 \dot{X}^\mu}{\sqrt{-(X')^2 \dot{X}^2}} \quad \Rightarrow \quad n\cdot\mathcal{P}^\tau = \frac{1}{2\pi\alpha'} \frac{(X')^2 (n\cdot\dot{X})}{\sqrt{-\dot{X}^2 (X')^2}}$$

but $n\cdot X = \beta\alpha'(n\cdot p)\tau$ so $(n\cdot\dot{X}) = \beta\alpha'(n\cdot p)$

Also, since $(n\cdot p)\sigma = \frac{2\pi}{\beta}\int_0^\sigma d\tilde{\sigma}\left[\mathcal{P}^\tau(\tau,\tilde{\sigma})\cdot n\right]$

$$(n\cdot p) = \frac{2\pi}{\beta} n\cdot\mathcal{P}^\tau \quad \text{so}$$

$$n\cdot\mathcal{P}^\tau = \frac{\beta}{2\pi}(n\cdot p) = \frac{n\cdot\dot{X}}{2\pi\alpha'} = \frac{1}{2\pi\alpha'}\frac{(X')^2\, n\cdot\dot{X}}{\sqrt{-\dot{X}^2 (X')^2}}$$

(4.2) so $\Rightarrow \dfrac{(X')^2}{\sqrt{-\dot{X}^2 (X')^2}} = 1 \Rightarrow \boxed{\dot{X}^2 + (X')^2 = 0}$

Combining (4.1) & (4.2) gives $\boxed{(\dot{X} \pm X')^2 = 0}$

Momenta simplify:

(4.3) $$\boxed{\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu}$$

and

(4.4) $$\boxed{\mathcal{P}^{\sigma\mu} = \frac{-1}{2\pi\alpha'} X'^\mu}$$

(4.5) and so $\boxed{\ddot{X}^\mu - (X^\mu)'' = 0}$ equation of motion

The solution to the e-o-m (4.5) can be Fourier-expanded with the centre of mass motion split off:

(4.6a) $$\boxed{X^\mu_{\text{open}}(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\,\alpha_0^\mu\,\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\sigma)}$$

standing waves satisfy Neumann BC's @ endpoints.

where

(4.6b) $$\boxed{\alpha_0^\mu \equiv \sqrt{2\alpha'}\,p^\mu}$$

so that $(\dot{X}^\mu \pm X'^\mu) = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-in(\tau\pm\sigma)}$

Light-cone gauge (specific case)

We set $n_\mu = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0\right)$

so that $n \cdot X = X^+$ so $n \cdot p = p^+$ and

(5.1) $\boxed{X^+(\tau,\sigma) = \beta\alpha' p^+ \tau}$, $\boxed{p^+ \sigma = \frac{2\pi}{\beta}\int_0^\sigma d\tilde{\sigma}\, \mathcal{P}^{\tau +}(\tau,\tilde{\sigma})}$

Then requiring $(\dot{X} \pm X')^2 = 0$

(5.2) $\Rightarrow$ $\boxed{(\dot{X}^- \pm X^{-\prime}) = \frac{1}{\beta\alpha'}\frac{1}{2p^+}(\dot{X}^i \pm X^{i\prime})^2}$

Using the general mode expansion (4.6a) and (5.1), see that

$x_0^+ = 0$, $\alpha_n^+ = \alpha_n^- = 0$; $p^+ = \sqrt{2\alpha'}\,\alpha_0^+$

(5.3) i.e. <u>in light-cone gauge, X^+ does not oscillate.</u>

Since we have (5.2), we can see that the $\ominus$ oscillators are going to be expressed in terms of quadratic combo's of X^i oscillators.

In detail, (see Zwiebach p.162), obtain [algebra]

(5.4) $\boxed{\sqrt{2\alpha'}\,\alpha_n^- = \frac{1}{2p^+}\sum_{p\in\mathbb{Z}} \alpha_{n-p}^I \alpha_p^I}$ (I is transverse)

So the α_n^- are <u>not</u> independent physical wiggles. It is the <u>X^I</u> that <u>have the "real" dynamics</u>. ☺

▷ Notice, for QFT students: when we set the gauge, (5.1), we got rid of <u>two</u> longitudinal degrees of freedom: not only did we lose X^+ oscillators via (5.1), the constraints knocked out X^- oscillators too. Quite generally, see that to knock out one longitudinal component of a vector, need to knock out a second. i.e. for A_μ in E&M also get only $(D-2)$ indep. d.o.f. (polarizations)

Virasoro modes

(6.1) Rename $$\boxed{L_n^\perp \equiv \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha^I_{n-p}\cdot\alpha^I_p}$$

(6.2) then $$\boxed{2p^+p^- = \frac{1}{\alpha'}L_0^\perp}$$

(6.3) so that $$\boxed{(\dot{X}^- \pm X^{-\prime}) = \frac{1}{p^+}\sum_{n\in\mathbb{Z}} L_n^\perp e^{-in(\tau\pm\sigma)}}$$

We can also easily compute the mass in light-cone gauge:

$$\begin{aligned} M^2 = -p^\mu p_\mu &= +p^+p^- + p^-p^+ - p^Ip^I \\ &= \frac{1}{\alpha'}L_0^\perp - p^Ip^I \\ &= \frac{1}{\alpha'}\left[\frac{1}{2}\alpha_0^I\alpha_0^I + \sum_{n\in\mathbb{Z}^+}\alpha_n^{I*}\alpha_n^I\right] - p^Ip^I \end{aligned}$$

(6.4) $$\boxed{M^2 = \frac{1}{\alpha'}\sum_{n\in\mathbb{Z}^+} n\, a_n^{I*}a_n^I} \geq 0$$ classically

where $\alpha_n^\mu = a_n^\mu\sqrt{n}$, $n\geq 1$.

Later on, we will quantize the string.

Each mode will contribute a zero-point energy, and we have to figure out the effect on the (mass)² – once we know how to quantize!

First, we discuss the simpler cases of scalar, vector and tensor fields in light-cone gauge to get some warm-up practice with the canonical quantization procedure.

Then we will graduate to quantizing strings.

Spin-zero

The relativistic action for a scalar, which is the minimal modification of "L=T-V" consistent with Lorentz invariance, is

(7.1) $$\boxed{S = \int d^D x \left\{ -\frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 \right\}}$$

This has canonical momenta so that

$$\Pi_\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi} = -\partial_\mu\varphi$$

$$\Pi_0 = \partial_0\varphi \quad \text{and so}$$

$$\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 \qquad (\geq 0 \;\checkmark)$$

and $E = \int d^{D-1}x\,\mathcal{H}$ is conserved.

The equations of motion are

$$\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\right) = 0$$

(7.2) i.e. $$\boxed{\partial^\mu\partial_\mu\varphi = m^2\varphi}$$

- This has plane-wave solutions

$$\phi(t,\vec{x}) = a_p e^{-iE_p t + i\vec{p}\cdot\vec{x}} + a_p^* e^{iE_p t - i\vec{p}\cdot\vec{x}} \qquad (\phi\in\mathbb{C})$$

where

(7.3) $$\boxed{E_{\vec{p}} = \sqrt{\vec{p}^{\,2} + m^2}}$$

- Fourier transforming is convenient:

$$\phi = \int \frac{d^D x}{(2\pi)^D}\, e^{ip\cdot x}\,\phi(p)$$

(7.4) $$\Rightarrow \boxed{(p^2 + m^2)\,\phi(p) = 0} \qquad \forall p$$

$p^2 + m^2 = 0$ is called $\begin{cases}\text{"on-shell"} \\ \text{"on the mass shell"}\end{cases}$.

Classically $\phi(p)$ vanishes off-shell.

Let's now do it in light cone coordinates.

Write $\vec{x}_T = (x^2, \dots, x^D)$ transverse coords
and
Fourier transform only in $\left\{ \begin{matrix} \text{transverse} \\ p^+ \end{matrix} \right\}$ directions

(8.1) $\Rightarrow$ $$\phi(x^+, x^-, \vec{x}_T) = \int \frac{dp^+}{2\pi} \int \frac{d^{D-2}\vec{p}_T}{(2\pi)^{D-2}} \, e^{-ix^- p^+ + i\vec{x}_T \cdot \vec{p}_T} \, \phi(x^+, p^+, \vec{p}_T)$$

This satisfies

(8.2) $$\left[i\partial_+ - \frac{1}{2p^+}(m^2 + p^I p^I) \right] \phi(x^+, p^+, \vec{p}_T) = 0$$ first order

Quantization

Let's write the field as a sum of plane-waves with operator coefficients this time :-

(8.3) $$\boxed{\phi = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-iE_p t + i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{iE_p t - i\vec{p}\cdot\vec{x}} \right)}$$

In a box of volume $V = \prod_{i=1}^{D} L_i$, we require
(8.4) quantization $\boxed{p_i = \frac{2\pi n_i}{L_i}}$ (no sum)

Working through the algebra gives

$$S = \int dt \left(\dot{a}^*(t)\,\dot{a}(t) \cdot \frac{1}{2E_p} - \frac{1}{2}E_p \, a^*(t)\, a(t) \right)$$

(8.5) and $$E = \int dt \left(\frac{1}{2E_p} \dot{a}^* \dot{a} + \frac{1}{2} E_p a^* a \right)$$

Canonical momenta

$$p = \frac{\partial \mathcal{L}}{\partial \dot{a}} = \dot{a}^* \quad ; \quad p^* = \frac{\partial \mathcal{L}}{\partial \dot{a}^*} = \dot{a}$$

(8.6) and they satisfy $\boxed{\ddot{a} = -E_p^2 a}$

This has the solution

$$a(t) = a_p e^{-iE_pt} + a^\dagger_{-p} e^{iE_pt}$$

(9.1) so $\boxed{H = E_p (a_p^* a_p + a_{-p}^* a_{-p})}$

Oscillators must satisfy canonical commutation relations

$$[a_p, a_p^\dagger] = 1 \quad ; \quad [a_{-p}, a_{-p}^\dagger] = 1$$

These come about from replacing Poisson brackets with commutators. This is canonical quantization.

In general, we label our a's with labels p that distinguish them as separate modes, so in fact

(9.2) $$\boxed{[a_p, a_k^\dagger] = \delta_{p,k} \quad ; \quad [a_p, a_k] = 0 = [a_p^\dagger, a_k^\dagger]}$$

Modulo a zero-point energy (see later on!) we have

(9.3) $$E = \sum_{\vec{p}} E_p a_p^\dagger a_p \quad , \quad \vec{P} = \sum_{\vec{p}} \vec{p} \underbrace{a^\dagger_p a_p}_{\text{number operator for } \vec{p}\text{-th mode}}$$

What we have are simple harmonic oscillators.

Vacuum state (no particles): $|\Omega\rangle$

Build one-particle state with momentum $\vec{p}$ by using $a_p^\dagger |\Omega\rangle$

This has momentum $\vec{p}$ and energy E_p - by using (9.3).

▷ In light-cone gauge, one-particle states for this scalar field are specified by $(p^+, \vec{p}_T)$ $\because$ $p^-(p^+, \vec{p}_T)$.

(9.4) $\Rightarrow$ $\boxed{a^\dagger_{p^+, p_T} |\Omega\rangle \Leftrightarrow \text{1-particle states}}$

$\boxed{\text{Spin-one}}$

Relativistic notation combines $\vec{E}$ and $\vec{B}$ into

(10.1) $$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$$

and the dynamics is invariant under the gauge symmetry

(10.2) $$\boxed{A_\mu \to A_\mu + \partial_\mu \epsilon} = A_\mu + \delta A_\mu$$

(10.3) $$S = \int d^D x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu}\right) \qquad \text{(free field)}$$

(10.4) whereupon $\boxed{\partial_\mu F^{\mu\nu} = 0}$

Using (10.1), this becomes

(10.5) $$\boxed{\partial^2 A^\nu - \partial^\nu (\partial \cdot A) = 0}$$

Again, Fourier transform:

(10.6) $$A^\mu(x) = \int \frac{d^D p}{(2\pi)^D} e^{ip\cdot x} A(p)$$

(10.7) then $$p^2 A^\mu - p^\mu (p\cdot A) = 0 \qquad \text{(e-o-m)}$$

Gauge transformation takes $\delta A_\mu(x) = \partial_\mu \epsilon$
so in momentum space this is

(10.8) $$\delta A(p) = i p_\mu \epsilon(p)$$

Reality of $\epsilon \Rightarrow \epsilon(-p) = \epsilon^*(p)$

In light-front gauge :-
Dynamical variables $\{A^+(p), A^-(p), A^I(p)\}$

(10.9) Pick $\boxed{A_+(p) = 0}$ (can always find an ϵ to do this, for this one component of A^μ).

Then e-o-m says

(10.10) ⊕ $p^+(A\cdot p) = 0$ i.e. $\boxed{p\cdot A = 0 \Rightarrow A^- = \frac{1}{p^+} p^I A^I}$

Ⓘ $\boxed{p^2 A^\mu = 0}$

Therefore, $A^I = 0$ off-shell. on-shell: $p^2 = 0$

(11.1) $$\boxed{\text{One-photon states : } \sum_{I=2}^{D-1} \xi_I \, a^{I\dagger}_{p^+, \vec{p}_T} |\Omega\rangle}$$

The ξ^I is called the polarization vector.

Spin-two

Einstein's equations for GR, $G_{\mu\nu} = 0$, can be linearized, as can the general coordinate transformations.

The eqn of motion becomes $(g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu})$

(11.2) $$\boxed{p^2 h^{\mu\nu} - p_\alpha (p^\mu h^{\nu\alpha} + p^\nu h^{\mu\alpha}) + p^\mu p^\nu h^\lambda{}_\lambda = 0}$$

The gauge transformations are

(11.3) $$\boxed{\delta h^{\mu\nu}(p) = i p^\mu \epsilon^\nu(p) + i p^\nu \epsilon^\mu(p)}$$

Light-cone gauge Variables are $\{h^{++}, h^{+-}, h^{--}, h^{+I}, h^{-I}, h^{IJ}\}$

By suitable gauge transformation, can set

$$\boxed{h^{++} = h^{+-} = h^{+I} = 0} \qquad (\text{like } A^+ = 0)$$

and the d.o.f. remaining are (h^{--}, h^{-I}, h^{IJ}).

Find

$$\boxed{h^{II} = 0} \quad (\text{traceless})$$

and

$$\boxed{p^2 h^{\mu\nu} = 0} \quad ; \quad \boxed{p_\alpha h^{\nu\alpha} = 0}$$ transverse

D.o.f. : symmetric traceless tensor

$\# = \frac{1}{2}(D-2)(D-1) - 1 = \frac{1}{2} D (D-3)$

Construct

$$\boxed{\text{One-graviton states : } \sum_{I,J=2}^{D-1} \xi_{IJ} \, a^{IJ\dagger}_{p^+, \vec{p}_T} |\Omega\rangle \; ; \; \xi_{II} = 0}$$

ξ_{IJ} is symmetric traceless $\Rightarrow n(D)$ d.o.f. on-shell.