TUTORIAL 8 - MIDTERM SOLUTIONS

$\boxed{Q1}$ A. $N=100$ COINS, 10 MORE HEADS THAN TAILS.

$\hookrightarrow N_\uparrow - N_\downarrow = 10 \equiv 2s$

$$\Rightarrow P(N,s) = g(N,0)\, e^{-2s^2/N}$$

$$\hookrightarrow \frac{P(N=100,5)}{P(N=100,0)} = \frac{\cancel{g(N,0)}\, e^{-2(5)^2/100}}{\cancel{g(N,0)}} = e^{-2\cdot 25/100} = e^{-1/2} = 0.61$$

$= 61\%$

$\therefore$ $\boxed{B}$ ANSWER.

$$\Rightarrow \frac{P(N,s)}{P(N,0)} = \frac{\cancel{N!}}{(\frac{1}{2}N+s)!\,(\frac{1}{2}N-s)!} \cdot \frac{(\frac{N}{2}!)^2}{\cancel{(N!)}} = \frac{(50!)\cdot(50!)}{(55)!\cdot(45)!}$$

$$= \frac{50\cdot 49\cdot 48\cdot 47\cdot 46}{55\cdot 54\cdot 53\cdot 52\cdot 51} = 0.61 \checkmark$$

$\boxed{Q1}$ B. $\boxed{pV = Nk_BT}$

(A) NO $\rho(n)$ $\rightarrow$ PHOTONS NOT IDEAL GAS. (NEVER CLASSICAL). ✗

(B) TEMPERATURE NOT TOO SMALL. $(\Delta\epsilon \ll \tau)$ ✓

(C) DISTINGUISHABILITY ✗ [redacted] ✗

(D) USED $p = -\left(\frac{\partial F}{\partial V}\right)_\tau$ ✓

$\therefore$ $\boxed{(E)}$ IS THE CORRECT ANSWER.

TUTORIAL 8 - MIDTERM SOLUTIONS CONT'D

Q1 c. μ.

(A) $S_1 \xleftrightarrow[\text{NOT } N]{\epsilon} S_2 \Rightarrow$ NO DEF'N OF $\mu \Rightarrow \mu_1 \neq \mu_2$ FOR EQUILIBRIUM ✗

(B) μ : $U_{GRAV} \leftrightarrow$ ACID pH IN BATTERY ??? ✗

(C) $\mu = \mu_{INT} + \mu_{EXT}$ = CONSTANT IN PLANETARY ATMOSPHERES ✗

[ONLY IF TEMPERATURE IS CONSTANT!]

(D) μ ~ MACROSCOPIC CHEMISTRY CONCEPT $\longleftarrow$ MICROSCOPIC PHYSICS ✓

Z, $F = -\tau \ln Z$, $\mu = \left(\frac{\partial F}{\partial N}\right)_\tau$

(E) 1^{ST} LAW : $dU = \tau d\sigma - p dV + \mu dN$ $\leftarrow$ NOT $d\mu$ ✗

$\therefore$ (D) IS ANSWER.

TUTORIAL 8 - MIDTERM SOLUTIONS CONT'D

Q2 CYANOGEN $\longrightarrow$ 3 STATES @ $E_0 = 4.7 \times 10^{-4}\,eV \times 1.6 \times 10^{-19}\,\frac{J}{eV}$
$\longrightarrow$ 1 STATE @ 0.

(A) NUMBER OF CYANOGEN IN GROUND : NUMBER IN E_0 = 10 : 3

$$\Rightarrow \frac{P(E_0)}{P(\text{GROUND})} = \frac{3 \cdot e^{-E_0/\tau}}{1 \cdot e^{0}} = \frac{3}{10}$$

$$\hookrightarrow \tau = \frac{E_0}{\ln 10} \Rightarrow \boxed{T = \frac{E_0}{k_B \ln 10} = 2.4\ K}$$

$$\simeq T_{CMB}.$$

(B) $$Z_1 = \sum_{\text{STATES}} e^{-E_s/\tau} = (1)\, e^{0} + (3) \cdot e^{-E_0/\tau} = 1 + 3e^{-E_0/\tau}$$

$$\hookrightarrow \boxed{Z_N = \frac{\left(1 + 3e^{-E_0/\tau}\right)^N}{N!}}$$

$\leftarrow$ NON-INTERACTING.

$\leftarrow$ INDISTINGUISHABLE.

(C) $$U \equiv \tau^2 \frac{\partial}{\partial \tau} \log Z_N = \tau^2 \frac{\partial}{\partial \tau}\left[N \log\left(1 + 3e^{-E_0/\tau}\right) - \log N!\right]$$

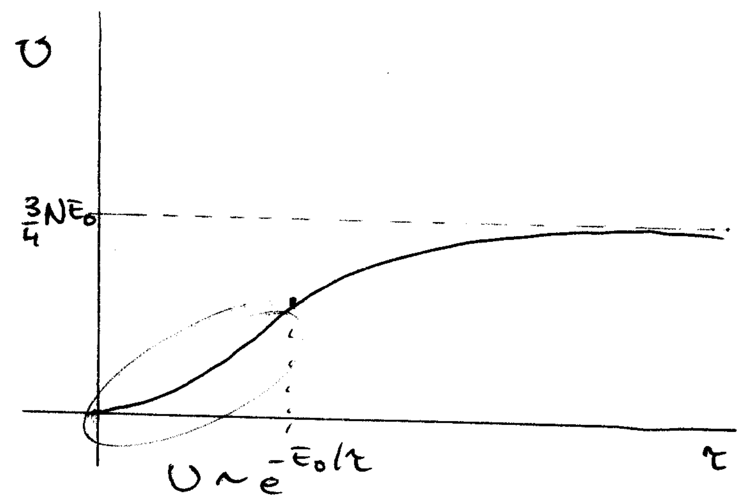
$$\boxed{U = 3NE_0 \frac{e^{-E_0/\tau}}{1 + 3e^{-E_0/\tau}}} = \frac{3NE_0}{e^{E_0/\tau} + 3}$$

$$\Rightarrow C_v \equiv \left(\frac{\partial U}{\partial \tau}\right)_v = (-1)(3NE_0) \frac{1}{\left(3 + e^{+E_0/\tau}\right)^2} e^{E_0/\tau} \left(-\frac{E_0}{\tau^2}\right)$$

$$\boxed{C_v = 3N\left(\frac{E_0}{\tau}\right)^2 \frac{e^{E_0/\tau}}{\left(e^{E_0/\tau} + 3\right)^2}}$$

TUTORIAL 8 - MIDTERM SOLUTIONS CONT'D

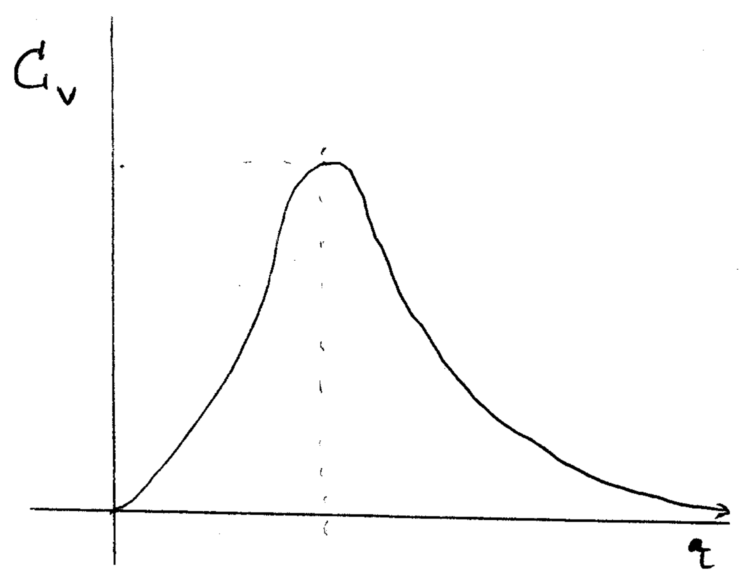
$\boxed{\text{Q2}}$ c. CONT'D.



$\Rightarrow$ AT LOW-τ, $U \simeq (3NE_0)e^{-E_0/\tau}$

$\Rightarrow$ AT HIGH-τ, $U \simeq \frac{3}{4}NE_0$

$\Rightarrow$ COMPARED TO THE SIMPLE TWO STATE SYSTEM, THE HIGH TEMP. ENERGY U IS GREATER ($U_{2\text{-STATE}} \rightarrow \frac{1}{2}NE_0$) AND AT LOW TEMP. THE SYSTEM OF CYANOGEN IS MORE LIKELY TO BE IN AN EXCITED STATE.



$\Rightarrow$ THERE IS A "BUMP" IN THE HEAT CAPACITY.

$\Rightarrow$ AT LOW-τ, $C_V \sim e^{-E_0/\tau}$

$\Rightarrow$ AT HIGH-τ, $C_V \rightarrow 0$.

Tutorial 8 – Midterm Solutions cont'd

Q3 A. $J_U = \left(\frac{\pi^2}{60}\right) \frac{k_B^4 T^4}{\hbar^3 c^2} \equiv \sigma_B T^4$ (1)

First, σ_B has <u>no</u> T-dependence $\Rightarrow$ all T-dependence is explicit in the Stefan–Boltzmann law.

Now, the units of J_U are:

$$[J_U] = \frac{J}{t \cdot L^2}$$

so we know that σ_B has <u>dimensions</u>. We are dealing with photons, so possible dimensionful constants are $c\ [\frac{L}{t}]$, $k_B\ [J/K]$ and $\hbar\ [J \cdot t]$. In order to cancel the K^4-dependence on the right side of (1) we therefore need

$$\sigma_B \sim k_B^4 \ldots$$

However this leaves us with

$$[k_B^4 T^4] = \frac{J^4}{\cancel{K^4}} \cdot \cancel{K^4} = J^4$$

units on the RHS. We only need one unit of energy, so we can use

$$\sigma_B \sim \frac{k_B^4}{\hbar^3} \Rightarrow \left[\frac{k_B^4 T^4}{\hbar^3}\right] = \frac{J}{t^3}$$

Lastly, then, we can use c to make the units of (1) agree:

$$\sigma_B \sim \frac{k_B^4}{\hbar^3 c^2} \Rightarrow [\sigma_B T^4] = \frac{J^{\cancel{4}}/\cancel{K^4} \cdot \cancel{K^4}}{\cancel{J^3} t^{\cancel{3}} \cdot L^2/\cancel{t^2}} = \frac{J}{t \cdot L^2}$$

Tutorial 8 – Midterm Solutions cont'd

Q3 A. cont'd

So, up to a dimensionless number ($\pi^2/60$) we know

$$\sigma_B \sim \frac{k_B^4}{\hbar^3 c^2}.$$

If we include the emissivity e, in general e is a function of frequency: $e(\omega)$, so when we integrate over frequencies to get J_U, the effect could be complex + dramatic. So we cannot use dimensional analysis in this case.

TUTORIAL 8 – MIDTERM SOLUTIONS CONT'D

Q3 CONT'D

B. $\frac{C_V}{3N} = \frac{4\pi^4}{5} k_B \left(\frac{T}{Q_D}\right)^3$

$$[C_V] = \left[\left(\frac{\partial U}{\partial T}\right)_V\right] = \frac{J}{K}$$

↳ $\left[\frac{C_V}{3N}\right] = \frac{J}{K}$ → ONLY CONSTANT WITH THESE UNITS IS k_B.

↳ $\frac{C_V}{3N} \sim k_B$ UP TO DIMENSIONLESS QUANTITIES.

⇒ DIM'LESS QUANTITY ACTUALLY CONTAINS

$\left(\frac{T}{\theta_D}\right)^{\#}$ ⇝ # NOT FIXED BY DIM'NAL ANALYSIS!

↳ THINKING OF U FOR PHONONS IN THREE DIMENSIONS,

EXPECT $U \sim T^4$ (FROM 3D : $\int d\Omega = \int dn\, n^2 \ldots$)

SO $C_V \sim T^3$ ⇝ EXPECT $\# = 3$.