"Quantum" Fermi Gas

Now we'll study what happens when gases (fermi first, then Bose) are <u>non</u>-classical.

We had a condition before for classical behavior

$$n << n_Q(\tau) = \left(\frac{M\tau}{2\pi\hbar^2}\right)^{3/2}$$ (monatomic)

So Quantum gas when $\boxed{n \gtrsim n_Q(\tau)}$

If $n = \frac{N}{V}$ is fixed for a particular problem,

we can turn this around and say

$$\underbrace{\frac{n^{2/3} \cdot 2\pi\hbar^2}{M}}_{= \tau_0} \gtrsim \tau \qquad \text{i.e.} \qquad \tau \leq \tau_0$$

So for <u>low</u> enough temperature any gas is quantum.

<u>Numbers?</u> Higher-level physics is <u>not</u> just about manipulating symbols, you need to have a sense of orders of magnitude ($\to$ "seat-of-the-pants intuition")

Suppose we look at electrons (!) with $m_e \cong 9.1 \times 10^{-31}$ kg

$$\Rightarrow \quad \tau_0 = \frac{2\pi\hbar^2 n^{2/3}}{M} \cong \left(7.67 \times 10^{-38}\right) n^{2/3}$$

Now suppose we had

$N_{av} \cong 6 \times 10^{23}$ atoms (1 mol.)

in 1L volume

$\Rightarrow \tau_0 \cong 5.5 \times 10^{-22}$ J

i.e. $T_0 = \dfrac{\tau_0}{k_B} \cong 40$ K pretty cold

For fixed N need

$$\left(\frac{N}{V}\right)^{2/3} \frac{2\pi\hbar^2}{M} \frac{1}{\tau} \gtrsim 1 \quad \text{for quantum gas}$$

$$\frac{N^{2/3}\, 2\pi\hbar^2}{M} \gtrsim \tau V^{2/3}$$

To raise τ, scrunch down V

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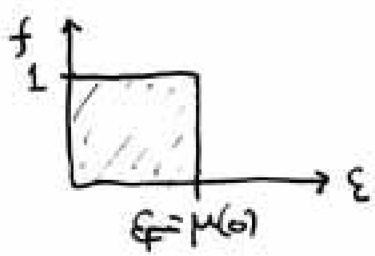
Remember our distribution function for fermions -

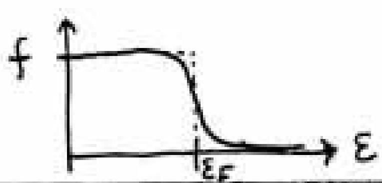
$$f_{FD}(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/\tau} + 1}$$

At $\tau = 0$,



i.e. all orbitals with $\varepsilon < \mu(0) \equiv \varepsilon_F$ are filled, all above ε_F empty.

For $\dfrac{\tau}{\varepsilon_F} \ll 1$,



small leakage above ε_F

Degenerate Fermi gas -

when $\tau \ll \varepsilon_F = \mu(0)$

Then orbitals up to ε_F almost all full,
orbitals a little bit above ε_F also filled.

Relevance In Real Life?

- Conduction electrons in metals
- white dwarf stars

} fermion is 'elementary' electron

- liquid ^3He
- nuclear matter

} proton & neutron are composite fermions
made of quarks (up and down)

In all these cases,
the relevant fermion we'll deal with has spin-half.

The Pauli Exclusion Principle does not allow 2 fermions in same quantum state

So for any orbital of given energy, the only way to have >1 fermion at a given energy is to have spins different

i.e. 2 orbitals $\begin{cases} @ \text{ energy } \varepsilon, \text{ spin-up } \uparrow \\ @ \text{ energy } \varepsilon, \text{ spin-down } \downarrow \end{cases}$

Groundstate of 3-d Fermi Gas

Particle Number:

Previously we worked out the energy levels of a particle in a box using the Schrödinger eqn

$$\varepsilon = \frac{\hbar^2}{2M}\left(\frac{\pi}{L}\right)^2 (\vec{n}^2)$$

$$\vec{n}^2 = n_x^2 + n_y^2 + n_z^2$$

Let's find the $|\vec{n}|$ that corresponds to the Fermi energy $\varepsilon_F \equiv \mu(0)$. At $\tau=0$, all orbitals up to ε_F are filled, all above empty.

→ Fill up to N particles.

* Let's use the approx that

$$\sum_{\ldots}(\ldots) \longrightarrow \int_{\ldots}(\ldots)$$

We have $\sum_n (2) = N$

$$\rightarrow N = \int_0^{?} dn_x \int_0^{?} dn_y \int_0^{?} dn_z$$

$$\equiv \underbrace{\frac{1}{8}}_{\text{first octant}} \cdot \underbrace{2}_{m_s = \pm\frac{1}{2}} \cdot \int_0^{n_F}\int_0^{4\pi} n^2 \, dn \, d\Omega$$

$$= \frac{1}{4} \cdot 4\pi \cdot \left(\frac{n_F^3}{3}\right) = \frac{\pi}{3} n_F^3 \Rightarrow \boxed{n_F = \left(\frac{3N}{\pi}\right)^{\frac{1}{3}}}$$

So

$$\varepsilon_F = \frac{\hbar^2}{2M}\frac{\pi^2}{L^2} n_F^2 = \frac{\hbar^2}{2M}\left(\frac{\pi n_F}{L}\right)^2$$

$$= \frac{\hbar^2}{2M}\left\{\frac{\pi}{L}\left(\frac{3N}{\pi}\right)^{1/3}\right\}^2$$

$$= \frac{\hbar^2}{2M}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

$$\boxed{\varepsilon_F = \frac{\hbar^2}{2M}\left(3\pi^2 n\right)^{2/3}}$$

<u>Total energy?</u>

Let's call $U(\tau=0) \equiv U_0$.

Then

$$U_0 = 2\sum_{n \le n_F} \varepsilon_{\vec{n}}$$

not the concentration; n here is $|\vec{n}|$

$$= 2\cdot\frac{1}{8}\cdot\int_0^{n_F}\int_0^{4\pi} n^2\,dn\,d\Omega\left\{\frac{\hbar^2}{2M}\left(\frac{\pi n_F}{L}\right)^2\right\}$$

$$= \frac{\pi^3}{2M}\left(\frac{\hbar}{L}\right)^2 \underbrace{\int_0^{n_F} n^4\,dn}_{=\frac{n_F^5}{5}}$$

$$\Rightarrow \quad U_0 = \frac{\pi^3}{10M}\frac{\hbar^2}{L^2}n_F^5 = \frac{\pi^3\hbar^2}{10ML^2}\left\{\underbrace{\left(\frac{3N}{\pi}\right)}_{n_F^3} n_F^2\right\} = \frac{3\pi^2\hbar^2 N}{10ML^2}n_F^2$$

i.e. $\boxed{U_0 = \frac{3}{5}N\varepsilon_F}$

This means that the average energy per particle is

$$\frac{U_0}{N} = \frac{3}{5}\epsilon_F = \frac{3}{5}\left[\frac{\hbar^2}{2M}\left(3\pi^2\frac{N}{V}\right)^{2/3}\right]$$

this per-particle energy decreases with volume V for fixed N

In other words, it works against a tendency to bind

This phenomenon is sometimes referred to in terms of a degeneracy pressure.

* Applications to collapsed stars

  - white dwarfs where the fermions doing the "pushing" are electrons

  - neutron stars where the fermions in question are neutrons

In both cases, the gravitational attraction provides the binding ... equilibrium between these 2 influences yields a stable object

We looked at groundstate of 3-d Fermi gas.
We found, filling up orbitals to n_F,

$$n_F = \left(\frac{3N}{\pi}\right)^{\frac{1}{3}}$$

$$\varepsilon_F = \frac{\hbar^2}{2M} \frac{\pi^2 n_F^2}{L^2} = \frac{\hbar^2}{2M} (3\pi^2 n)^{2/3}$$

$n = \frac{N}{V}$

Total energy? $U(\tau=0) = U_0$

$$U_0 = \frac{3}{5} N \varepsilon_F$$

(! note, energy even at absolute zero temperature!)

Energy per particle

$$\frac{U_0}{N} = \frac{3}{5}\left[\frac{\hbar^2}{2M}\left(3\pi^2 \frac{N}{V}\right)^{2/3}\right]$$

increases as V decreases

"Fermions have elbows" :)

That $\left[\frac{\partial}{\partial V}\left(\frac{U_0}{N}\right)\right]_N < 0$ means that

the Pauli Exclusion Principle works <u>against</u> binding

This outward pressure

holds white dwarfs & neutrons stars up!

Fermions: electrons (white dwarfs), neutrons (neutron stars)

White Dwarfs

Typical white dwarf has

mass $\sim M_{sun} \sim 2\times10^{30}$ kg ← Sirius B

radius $\sim 1\% R_{sun} \sim 2\times10^{7}$ m

Then average density is

$$\rho_{w.d.} \sim \frac{M_{w.d.}}{\frac{4}{3}\pi R^3_{w.d.}} \sim 0.6\times10^{8}\ \text{kg/m}^3$$

$$= 0.6\times10^{5}\ \text{g/cm}^3 \quad \text{HUGE!}$$

Let us suppose white dwarf is collapsed Hydrogen - for sake of argument.

Mass of $^1H \doteq 1.67\times10^{-27}$ kg

Suppose $\rho_{w.d.}$ can be thought of as $\frac{m(^1H)}{\text{vol}(^1H)}$

then 0.6×10^{8} kg/m^3

translates to about 3×10^{-36} m^3 for a ^1H

So on average the spacing between 'H's is $\sim 10^{-12}$ m

which is smaller than the Bohr radius

$\Rightarrow$ electrons no longer "belong" to any given 'H,

they're all squished (ferociously squished)!

Compare also to Compton wavelength for e^-

$$\lambda_c(e^-) = \frac{h}{m_e c} \cong 2.4 \times 10^{-12}\ \text{m}$$

For an electron density of

$$n = \frac{N}{V} \cong \frac{1}{10^{-36}\,\text{m}^{3}} \sim 10^{+36}\,\text{m}^{-3}$$

$$\Rightarrow \varepsilon_F = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3}$$

$$\cong 3 \times 10^5\ \text{eV} = 300\ \text{keV}$$

Let's compare to electron mass!

$m_e = 511\ \text{keV}/c^2$ i.e. rest energy is 511 keV

So we are beginning to feel relativity (!)

Temp? Fermi temperature $T_F \equiv \dfrac{\varepsilon_F}{k_B} \cong 3 \times 10^9\ \text{K}$ FROM

c.f. thermal temperature $\sim 10^7$ K EXPT