⇒ possible occupation # for an orbital

fermions	0 or 1
bosons	0, 1, ...

Thermodynamics

Interested in thermal average properties.
If average occupancy f of an orbital is $|f| \ll 1$ then the availability of $n = 2, 3, \ldots$ won't make much difference to the thermal physics.

⇒ at low f we expect not to care whether we have fermions or bosons.
In other regions of parameter space it will matter

$\boxed{\text{FERMI-DIRAC distribution function}}$

Consider thought experiment
- * system = $\underline{\underline{1}}$ orbital, energy ε
  Temp τ, chem. pot'l μ
- * reservoir = all the other orbitals.

Now let's find the partition fn for fermion

example: $\bar{e}$ electron

Have τ and μ ⇒ need $\mathfrak{z}$

$$z = \sum_{N,s} e^{-\frac{(\varepsilon_{s(N)} - \mu N)}{\tau}}$$

For $N=0$: no particle, energy zero $\Rightarrow \varepsilon_0 = 0$

For $N=1$: 1 particle, energy $\varepsilon_1 \equiv \varepsilon$

$\Rightarrow$
$$z(\mu,\tau) = e^{-\frac{(\varepsilon-\mu)}{\tau}} + 1$$

Average value of N ?

$$\langle N \rangle = 0.\,P(0) + 1.\,P(1)$$
$$= \frac{e^{-\frac{(\varepsilon-\mu)}{\tau}}}{1 + e^{-\frac{(\varepsilon-\mu)}{\tau}}}$$

Define $\boxed{f(\varepsilon) \equiv \langle N(\varepsilon) \rangle}$ "filling fraction"

warning – this phrase used differently elsewhere

Then

$$\boxed{f(\varepsilon)_{FD} = \frac{1}{\left[e^{(\varepsilon-\mu)/\tau} + 1\right]}}$$

Fermi-Dirac distribution function

– for fermions

Terminology

In condensed matter physics μ has a special name

$$\mu \Leftrightarrow \text{"Fermi level"}$$

For the special case of $\tau = 0$

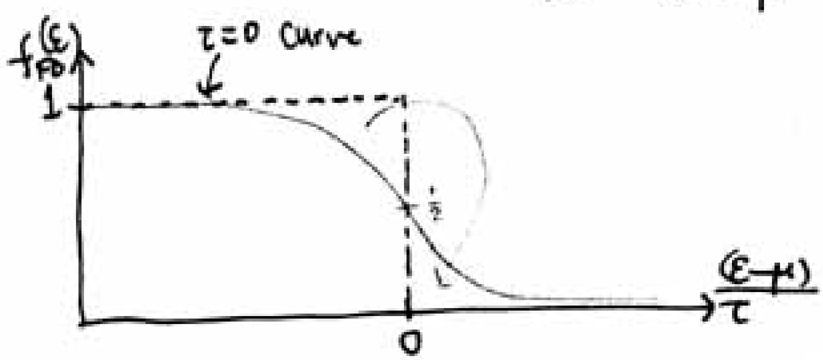
$$\mu(\tau=0) \equiv \varepsilon_F \quad \text{"Fermi energy"}$$

Let's take a look at the FD distribution fn!

$$f(\varepsilon)_{FD} = \frac{1}{\left[e^{(\varepsilon-\mu)/\tau} + 1\right]}$$

- When $\varepsilon = \mu$, no matter τ, $f = \frac{1}{2}$.
- For $\tau \to 0$, $\exp \to \begin{cases} +\infty & \text{if } \varepsilon > \mu \\ 0 & \varepsilon < \mu \end{cases}$
- For $\tau > 0$, $\exp \to \begin{cases} +\infty & \text{for } \varepsilon \gg \mu \\ 0 & \text{for } \varepsilon \ll \mu \end{cases}$

$\Rightarrow$



At $\tau > 0$, fermions previously in $\varepsilon < \mu$ $\to$ in $\varepsilon > \mu$

Example is electrons in metal
(conduction e^-'s).

BOSE-EINSTEIN distribution function

Examples of bosons -

- photons
- phonons
- ^4He

Now let's compute $f(\varepsilon)$ for BOSONS.

We need to find $\mathfrak{z}$ first.

Use same assumptions as for fermion setup - except (of course) that $N = 0, 1, \dots$ up to ∞

So.

$$\mathfrak{z}(\mu,\tau) = \sum_{N,s} e^{-(\varepsilon_{s(N)} - \mu N)/\tau}$$

$$= 1 + e^{-(\varepsilon-\mu)/\tau} + e^{-(2\varepsilon - 2\mu)/\tau} + \dots$$

$$= 1 + e^{-(\varepsilon-\mu)/\tau} + \left(e^{-(\varepsilon-\mu)/\tau}\right)^2 + \dots$$

$$= \frac{1}{1 - e^{-(\varepsilon-\mu)/\tau}}$$

(geometric series $\sum_{N=0}^{\infty} x^N = \frac{1}{1-x}$, $x<1$)

Now we want to find $\langle N(\varepsilon) \rangle$

$$\langle N \rangle = \tau \frac{\partial \log z}{\partial \mu}$$

Here, $z = \dfrac{1}{1 - e^{-(\varepsilon-\mu)/\tau}}$ so

$$\frac{\partial z}{\partial \mu} = \frac{1}{\tau} \cdot - e^{-(\varepsilon-\mu)/\tau} \left(\frac{-1}{1-e^{-(\varepsilon-\mu)/\tau}} \right)^2$$

$$= \frac{\frac{1}{\tau} e^{-(\varepsilon-\mu)/\tau}}{\left[1 - e^{-(\varepsilon-\mu)/\tau}\right]^2}$$

thus $\langle N(\varepsilon) \rangle = \tau \left(1 - e^{-(\varepsilon-\mu)/\tau}\right) \frac{1}{\tau} \dfrac{e^{-(\varepsilon-\mu)/\tau}}{\left[1 - e^{-(\varepsilon-\mu)/\tau}\right]^2}$

i.e. $\langle N(\varepsilon) \rangle = \dfrac{e^{-(\varepsilon-\mu)/\tau}}{\left[1 - e^{-(\varepsilon-\mu)/\tau}\right]}$

or $\boxed{f(\varepsilon)_{BE} = \dfrac{1}{e^{(\varepsilon-\mu)/\tau} - 1}}$ Bose-Einstein distribution fn for BOSONS.

notice All the difference!

Now let's look at the behaviour of f_{BE}.

· Anything special at $\mu = \varepsilon$?

then $f_{BE} \to \infty$! oops!

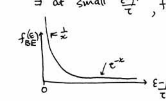
So $f_{BE}(\varepsilon) = \dfrac{1}{e^{(\varepsilon-\mu)/\tau} - 1}$ has a maximum value (∞) as $\varepsilon \to \mu$ from above.

How does it blow up?

Consider $x \equiv \frac{(\varepsilon-\mu)}{\tau}$. Then $\varepsilon \to \mu$ is $x \to 0$

So $e^x \xrightarrow[x\to 0]{} 1 + x + \mathcal{O}(x^2)$

$\Rightarrow$ at small $\frac{\varepsilon-\mu}{\tau}$, $f_{BE}(\varepsilon) \to \dfrac{1}{(\varepsilon-\mu)/\tau} = \dfrac{\tau}{(\varepsilon-\mu)} \xrightarrow[\text{as}]{} \infty$ $\frac{1}{x}$.



* Notice $f_{FD}(\varepsilon) = \dfrac{1}{e^{(\varepsilon-\mu)/\tau} + 1}$, $f_{BE}(\varepsilon) = \dfrac{1}{e^{(\varepsilon-\mu)/\tau} - 1}$

When $(\frac{\varepsilon-\mu}{\tau}) \gg 1$, the ± 1 are not important

and $f_{FD}(\varepsilon) \xrightarrow[\frac{\varepsilon}{\tau} \gg \frac{\mu}{\tau}]{} e^{-(\varepsilon-\mu)/\tau} \ll 1$ (ξ Gibbs factor)

$f_{BE}(\varepsilon) \longrightarrow e^{-(\varepsilon-\mu)/\tau} \ll 1$ $\boxed{\text{Classical limit } f \ll 1}$

So we have the table

	Fermion	Boson
Classical	$f_{FD} \ll 1$	$f_{BE} \ll 1$
Quantum	$f_{FD} \lesssim 1$	$f_{BE} \gg 1$

This is what gives rise to the phenomenon of Bose-Einstein Condensation

Recap:

occupation #s for orbital $\begin{cases} 0 \text{ or } 1 & \text{Fermion} \\ 0,1,2,\ldots & \text{Boson} \end{cases}$

$$\Rightarrow \boxed{f_{FD}(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/\tau}+1} \quad , \; f_{BE}(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/\tau}-1}}$$

We can summarize these as

$$f(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/\tau} \pm 1}$$

Classical limit $\Leftrightarrow$ low occupancy probability

$$\rightarrow f(\varepsilon) \rightarrow e^{-(\varepsilon-\mu)/\tau} \ll 1$$

$$= e^{-\varepsilon/\tau} \cdot \lambda$$

activity, previously defined as $\lambda \equiv e^{\mu/\tau}$.

The result $\boxed{f(\varepsilon) \rightarrow \lambda e^{-\varepsilon/\tau}}$ is called the Classical distribution function

Ideal Gas (again)

Let's use our new knowledge to connect back to several aspects of classical ideal gas physics that we derived previously in a more basic way.

Particle #

We know that

$\langle N \rangle$ average # particles is sum over states.

Here, it's $= \sum_s f(\varepsilon_s)$

$$= \sum_s e^{\mu/\tau} e^{-\varepsilon_s/\tau}$$

$$= e^{\mu/\tau} \underbrace{\sum_s e^{-\varepsilon_s/\tau}}$$

but this is just the partition fn (regular, not grand!) for a single molecule of ideal gas!

So $\langle N \rangle = \lambda Z_1$

Ages ago we had, from working out energy levels in a box, $Z_1 = \left(\frac{M\tau}{2\pi\hbar^2}\right)^{3/2} V = n_Q(\tau) V$

so $\langle N \rangle = \lambda n_Q V$ or $\lambda = \frac{n}{n_Q} = e^{\mu/\tau}$

i.e. $\mu = \tau \log(n/n_Q)$

* What changes if the zero of energy is shifted by an amount $+\delta$?

Then $\langle N \rangle = \sum_s f(\varepsilon_s)$

$$= e^{\mu/\tau} \sum_s e^{-(\varepsilon_s + \delta)/\tau}$$

$$= e^{-\delta/\tau} \left(e^{\mu/\tau} \sum_s e^{-\varepsilon_s/\tau} \right)$$

so now $e^{(\mu - \delta)/\tau} = \dfrac{\langle N \rangle}{Z_1}$

$$\Rightarrow \mu \to \tau \log \left({}^{n}/_{n_Q} \right) + \delta$$

Chemical potential gets shifted up accordingly. This is in tune with its interpretation as a "real" energy (which we discussed ~ lead-acid battery part).

As before, connect to F, p, U, $N\tau$, etc... and find ideal gas equation $pV = N\tau$

and $U = \frac{3}{2} N\tau$

this number comes from fact that live in 3 (space) dimensions

For classical gas (ideal)
in d (space) dimensions,
would find $U = \frac{d}{2} N\tau$

or $\frac{1}{2} N\tau$ per "degree of freedom"

Equipartition Theorem

in ideal gas, each independent mode of motion that's accessible to an ideal gas molecule (in the sense that the mode is not too costly in energy @ $\tau = k_B T$) gives a "degree of freedom" and an energy $\frac{1}{2} N\tau$

$\Rightarrow$ for F degrees of freedom

Internal energy $U = \frac{F}{2} N\tau$

What other degrees of freedom can a gas molecule have?