Last time-

we started on the Photon Gas.

- got partition function for single mode of EM field at frequency ω

$$Z_\omega = \frac{1}{(1-e^{-\hbar\omega/\tau})}$$

and avg photon #

$$\langle s \rangle = \frac{e^{-\hbar\omega/\tau}}{(1-e^{-\hbar\omega/\tau})} = \frac{1}{(e^{\hbar\omega/\tau}-1)} \qquad \underline{\text{Planck}}$$

Now we will

(a) Work out possible ω's for photons in boxes

(b) Sum over all modes $\Rightarrow Z_{total}$.

* Maxwell's Eqns (no ρ, $\vec{J}$)

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)$$

$$= \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E}$$

$$\Rightarrow \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right]\vec{E} = 0 \qquad \text{Wave Equation}$$

* Each EM mode has $\underline{2}$ independent polarizations ($\vec{\nabla}\cdot\vec{E}$ puts 1 condition on 3 vrbles)

Let us try standing waves

$$E_x = E_{x0} \sin(\omega t) \cos\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right)$$

$$E_y = E_{y0} \sin(\omega t) \sin\left(\frac{n_x \pi x}{L}\right) \cos\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right)$$

$$E_z = E_{z0} \sin(\omega t) \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \cos\left(\frac{n_z \pi z}{L}\right)$$

Then the wave equation says

$$\vec{n}^2 = \frac{\omega^2 L^2}{c^2 \pi^2}$$

and Maxwell says

$$\vec{n} \cdot \vec{E}_0 = 0$$

So define $\omega_n = \frac{|\vec{n}| c \pi}{L}$.

Then the average energy is going to be

$$U = \sum_n \langle \varepsilon_n \rangle$$

$$= \sum_n \frac{\hbar \omega_n}{e^{\hbar \omega_n / \tau} - 1}$$

* Approximate the sum by an integral.

$\int d(\text{what})$??

- Integrate over n_x, n_y, n_z. Since ω_n depends only on $|\vec{n}|$, smartest to use spherical coords

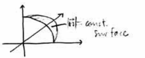
$\{\vec{n}\} \to \{n, \Omega\}$

$\llcorner$ solid angle : $\int d\Omega = \int_0^\pi d\theta \int_0^{2\pi} \sin\theta d\varphi$

If we had plane waves (L- or R-moving) we'd integrate over all $\vec{n}$

BUT we have standing waves, so we integrate over only $n_x, n_y, n_z \geq 0$!

$\Rightarrow$ first octant



$$\sum_n \to 2\left\{\frac{1}{8}\int_0^\infty dn \int_0^{4\pi} d\Omega\, n^2\right\}$$

measure of integration

$$\to \pi \int_0^\infty dn\, n^2$$

$$\therefore \quad U = \pi \int_0^\infty \frac{dn\, n^2 \left(\frac{\hbar c\pi}{L}\right) n}{e^{\hbar c\pi n/(L\tau)} - 1}$$

$$= \left(\frac{\hbar c\pi^2}{L}\right) \int_0^\infty \frac{dn\, n^3}{e^{(\hbar c\pi/L\tau)n} - 1}$$

Change variables to $w = \left(\frac{\hbar c\pi}{L\tau}\right)$

$$\therefore \quad U = \left(\frac{\hbar c\pi^2}{L}\right)\left(\frac{L\tau}{\hbar c\pi}\right)^4 \int_0^\infty \frac{dw\, w^3}{e^w - 1}$$

Consider $I_3 = \int_0^\infty \frac{dw\, w^3}{e^w - 1} = \int_0^\infty dw\, w^3 \frac{e^{-w}}{(1 - \underbrace{e^{-w}}_{\text{ranges from } 1^- \text{ to } 0.})}$

$$= \int_0^\infty dw\, w^3 \left(e^{-w} + e^{-2w} + e^{-3w} + \cdots\right)$$

$$= \int_0^\infty dw\, w^3 e^{-w} \left\{1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right\}$$

$$= \{6\} \qquad \left\{\zeta(4) = \frac{\pi^4}{90}\right\} \qquad = \pi^4/15$$

So, using $L^3 = V$,

$$\boxed{U = \frac{\pi^2}{15\hbar^3 c^3} V \tau^4}$$ Stefan-Boltzmann

Notes

(a) Extensive in volume V (✓)

(b) Proportional to τ^4

because of that change of variables in 3-dim integral

(in d-dim, $U \propto \tau^{d+1}$ (!))

[(c) dimensions

Consider photon of wavelength L

This has energy $E_L = \frac{hc}{\text{"}\lambda\text{"}} = \frac{hc}{L}$

so $\frac{U}{V} = \left(\frac{8\pi^5}{15}\right) \frac{\tau^4}{E_L^3}$]

Def Spectral density for energy, u_ω, defined by

$$\int u_\omega \, d\omega = \frac{U}{V}.$$

We had before that $U = \pi \int_0^\infty dn \, \frac{n^2 (\hbar\omega_n)}{e^{\hbar\omega_n/\tau} - 1}$

and $\omega_n = \frac{nc\pi}{L}$

So

$$U = \pi\hbar \left(\frac{L}{c\pi}\right)^3 \int_0^\infty d\omega \, \frac{\omega^3}{(e^{\hbar\omega/\tau}-1)}$$

$$\Rightarrow \boxed{u_\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/\tau}-1}}$$ spectral density for photons

"Planck Radiation Law"

for fixed τ

Let's find the maximum of this function

$$\frac{du_\omega}{d\omega} = \frac{\hbar}{\pi^2 c^3}\left\{ \frac{3\omega^2}{e^{\hbar\omega/\tau}-1} + \frac{\omega^3 \cdot -1}{(e^{\hbar\omega/\tau}-1)^2} \cdot e^{\hbar\omega/\tau} \cdot \frac{\hbar}{\tau} \right\}$$

$$= \frac{\hbar}{\pi^2 c^3} \frac{1}{(e^{\hbar\omega/\tau}-1)^2} \left[3\omega^2 (e^{\hbar\omega/\tau}-1) - \frac{\hbar\omega^3}{\tau} e^{\hbar\omega/\tau} \right]$$

$= 0$ @ extremum

Let $x = \frac{\hbar\omega}{\tau}$. Then at extremum

$$3x^2 (e^x - 1) = x^3 e^x$$

$$3(e^x - 1) = x e^x$$

$$x e^x - 3e^x + 3 = 0$$

Solve numerically $\Rightarrow x = 0$ (not interesting)

and $x \sim 2.82 \equiv b_W$

So the peak ω and τ satisfy

$$\hbar\omega_{peak} = \frac{\hbar c}{\lambda_{peak}} = b_\omega \tau_{peak} = b_\omega k_B T_{peak}$$

$$\text{so} \quad \lambda_{peak} T_{peak} = \frac{hc}{b_\omega k_B} = \text{constant}$$

Wien displacement law.

$\Rightarrow$ The hotter the photon gas, the bluer ☺

//

Entropy

We know that

$$dU = \tau d\sigma - p dV$$

so at constant volume

$$dU = \tau d\sigma$$

But $U = \frac{\pi^2}{15\hbar^3 c^3} V \tau^4$ so $dU\Big|_{\substack{const\\V}} = \frac{4\pi^2 V}{15\hbar^3 c^3} \tau^3 d\tau$

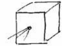
$$\Rightarrow \quad d\sigma = \frac{4\pi^2 V}{15\hbar^3 c^3} \tau^2 d\tau$$

so $\boxed{\sigma = \frac{4\pi^2 V}{45\hbar^3 c^3} \tau^3}$ Entropy scales like τ^3. (τ^d in d-dim. (!))

$$\left[\text{or} \quad \frac{\sigma}{V} = \frac{4\pi^2}{45} \left(\frac{\tau}{\hbar c}\right)^3 \quad \text{or} \quad \sigma = \frac{32\pi^5}{45} \left(\frac{L}{E_L}\right)^3 \right]$$

Black Body

One way is to have a cavity full of photons and make a wee hole:



Energy flux density

$J_u \equiv \dfrac{\text{rate of energy emission}}{\text{Unit area}}$

Units?

$$[J_U] = \frac{[\text{Energy}]}{[\text{time}][\text{area}]} = \frac{[\text{energy}]}{\frac{1}{c}[\text{Volume}]} = c\left[\frac{U}{V}\right]$$

(because photons move @ c!)

Spectral density of J_U, j_u,

arriving in $d\Omega$ is $\propto \cos\theta$

$$j_u = \underbrace{\int \frac{d\Omega}{4\pi}}_{\text{normalized}} \cos\theta \, (c u_\omega)$$

not a function of angles

but we integrate only from $\theta = 0$ to $\theta = \frac{\pi}{2}$ (otherwise we get a big fat zero)

$$\therefore j_u = \frac{c}{4\pi} u_\omega \int_0^{2\pi} d\varphi \sin\theta \int_0^{\pi/2} \cos\theta \, d\theta$$

$$= \frac{c u_\omega}{4}$$

so $J_u = \dfrac{c}{4}\dfrac{U}{V} = \dfrac{\pi^2 \tau^4}{60\hbar^3 c^2} \equiv \sigma_B T^4$

Stefan-Boltzmann const.

$(\sigma_B \equiv \pi^2 k_B^4 / (60\hbar^3 c^2))$

Example of blackbody

Most perfect BB spectrum in Nature is radiation in the sky left over from the big bang !!

Universe now has only microwave radiation left over from a very early epoch when everything was unbelievably hot.

$T \sim 3K$

(Discussion)

Absorption versus Radiation

Emissivity is defined as $J_u = e \sigma_B T^4$

$0 \leq e \leq 1$

What about absorptivity a ?

For a wee hole, ingoing radiation gets absorbed 100% efficiently = for the perfect Blackbody.
(the incoming stuff just goes into the box, gets bounced off walls, etc...)

Thermal equilibrium $\Rightarrow$ $a = e$ for any frequency
$\rightarrow$ for all frequencies.