Q1: Definitions

(a) Helmholtz Free Energy F:

$$F \equiv U - \tau\sigma$$

is the corresponding thermal potention to internal energy U, and is associated with isothermal processes. It minimizes energy and maximizes entropy. F is minimum for a system S in thermal contact with a reservoir R at constant volume.

(b) Ideal Gas: gas comprized of structureless non-interacting particles in the classical regime ($n/n_Q \ll 1$) ← which means the gas is rarified and at low pressure. The atmosphere of the Earth is taken as ideal gas.

(c) Equipartition of Energy: equal contribution of each "degree of freedom" to the partition function. $\frac{1}{2}\tau$ / degree of freedom $\Rightarrow U = \frac{3}{2}N\tau$ for ideal gas.

(d) Debye Temperature: a temperature in solid state physics which divides the low temperature quantum regime from the high temperature classical regime.

(e) Grand Partition Function: is a partition function which allows for the exchange of not only energy, but particles in a thermal system in contact with thermal reservoir.

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Q2: Energy fluctuations

Let $U \equiv \langle \varepsilon \rangle$ and $U = \tau^2 \frac{\partial}{\partial \tau}(\log Z) = \tau^2 \left(\frac{1}{Z} \frac{\partial Z}{\partial \tau} \right)$

Then
$$\left(\frac{\partial U}{\partial \tau} \right)_V = 2\tau \overbrace{\left(\frac{1}{Z} \frac{\partial Z}{\partial \tau} \right)} - \tau^2 \overbrace{\frac{1}{Z^2} \left(\frac{\partial Z}{\partial \tau} \right)^2} + \frac{\tau^2}{Z} \cdot \frac{\partial^2 Z}{\partial \tau^2} =$$

$$= 2\tau \cdot \frac{U}{\tau^2} - \tau^2 \cdot \frac{U^2}{\tau^4} + \frac{\tau^2}{Z} \cdot \frac{\partial^2 Z}{\partial \tau^2} =$$

$$= 2\tau \cdot \frac{\langle \varepsilon \rangle}{\tau^2} - \frac{\langle \varepsilon \rangle^2}{\tau^2} + \frac{\tau^2}{Z} \cdot \frac{\partial^2 Z}{\partial \tau^2}$$

and multiplying by τ^2 we get

$$\tau^2 \left(\frac{\partial U}{\partial \tau} \right)_V = 2\tau \langle \varepsilon \rangle - \langle \varepsilon \rangle^2 + \frac{\tau^4}{Z} \cdot \frac{\partial^2 Z}{\partial \tau^2}$$

where
$$Z = \sum_s \exp(-\varepsilon_s / \tau)$$

$$\therefore \frac{\partial Z(\tau)}{\partial \tau} = \sum_s \exp(-\varepsilon_s/\tau) \cdot [+\varepsilon_s/\tau^2] = \frac{1}{\tau^2} \sum_s \varepsilon_s \exp(-\varepsilon_s/\tau) =$$

$$= \frac{1}{\tau^2} Z \langle \varepsilon \rangle \equiv \frac{1}{\tau^2} Z U$$

$$\frac{\partial^2 Z(\tau)}{\partial \tau^2} = -\frac{2}{\tau^3} \sum_s \varepsilon_s \exp(-\varepsilon_s/\tau) + \frac{1}{\tau^2} \sum_s \varepsilon_s \exp(-\varepsilon_s/\tau) [+\varepsilon_s/\tau^2] =$$

$$= -\frac{2}{\tau^3} \sum_s \varepsilon_s \exp(-\varepsilon_s/\tau) + \frac{1}{\tau^4} \sum_s \varepsilon_s^2 \exp(-\varepsilon_s/\tau)$$

By definition

$$\langle \varepsilon \rangle = \frac{\sum_s \varepsilon_s \exp(-\varepsilon_s/\tau)}{Z}$$

$$\langle \varepsilon^2 \rangle = \frac{\sum_s \varepsilon_s^2 \exp(-\varepsilon_s/\tau)}{Z}$$

so that

$$\frac{1}{Z}\frac{\partial^2 Z(\tau)}{\partial \tau^2} = -\frac{2}{\tau^3}\frac{\sum_s \varepsilon_s \exp(-\varepsilon_s/\tau)}{Z} + \frac{1}{\tau^4}\frac{\sum_s \varepsilon_s^2 \exp(-\varepsilon_s/\tau)}{Z} =$$

$$= -\frac{2}{\tau^3}\langle \varepsilon \rangle + \frac{1}{\tau^4}\langle \varepsilon^2 \rangle \quad \checkmark$$

and

$$\tau^2\left(\frac{\partial U}{\partial \tau}\right)_V = 2\tau\langle \varepsilon \rangle - \langle \varepsilon \rangle^2 + \tau^4\left[-\frac{2}{\tau^3}\langle \varepsilon \rangle + \frac{1}{\tau^4}\langle \varepsilon^2 \rangle\right]$$

$$= \cancel{2\tau\langle \varepsilon \rangle} - \langle \varepsilon \rangle^2 - \cancel{2\tau\langle \varepsilon \rangle} + \langle \varepsilon^2 \rangle =$$

$$= \langle \varepsilon^2 \rangle - \langle \varepsilon \rangle^2$$

But

$$\langle(\varepsilon - \langle \varepsilon \rangle)^2\rangle = \langle(\varepsilon^2 - 2\varepsilon\langle \varepsilon \rangle + \langle \varepsilon \rangle^2)\rangle =$$

$$= \langle \varepsilon^2 \rangle - 2\langle \varepsilon \rangle\langle \varepsilon \rangle + \langle \varepsilon \rangle^2 = \langle \varepsilon^2 \rangle - \langle \varepsilon \rangle^2$$

and in this way we prove that

$$\boxed{\langle(\varepsilon - \langle \varepsilon \rangle)^2\rangle = \tau^2\left(\frac{\partial U}{\partial \tau}\right)_V = \tau^2 C_V} \quad \text{QED!}$$

Q3: One-Dimensional Paramagnet

(a) We have 3 possible spin components:

$$(\uparrow, 0, \downarrow) \Longleftrightarrow (+1, 0, -1)$$

For non-interacting particles the usual binomial expansion is upgraded to multinomial-(trinomial) expansion

$$(\uparrow + 0 + \downarrow)^N = (x+y+z)^N = (1+1+1)^N = \underline{\underline{3^N}}$$

with 3^N total # of states.

$$\therefore (x+y+z)^N = \sum_{N\uparrow} \sum_{N\downarrow} \sum_{N_0} \frac{N!}{N\uparrow! \, N\downarrow! \, N0!} x^{N\uparrow} y^{N\downarrow} z^{N0}$$

with constraints:

$$\begin{cases} (1.) & N\uparrow + N\downarrow + N0 = N \\ (2.) & N\uparrow - N\downarrow = S, \quad S = 0, \pm 1, \pm 2, \pm 3, \dots, \pm N \end{cases}$$

Solving (1.) + (2.) for $N\uparrow$ & $N\downarrow$ we get

$$N\uparrow = \frac{1}{2}(N - N0 + S)$$

$$N\downarrow = \frac{1}{2}(N - N0 - S)$$

so that the multiplicity function is

$$g(N, N0, S) = \frac{N!}{\left[\frac{1}{2}(N-N0) + \frac{S}{2}\right]! \left[\frac{1}{2}(N-N0) - \frac{S}{2}\right]! \, N0!}$$

Using Stirling approximation

$$\log g(N, N0, S) = \log N! - \log[\tfrac{1}{2}(N-N0) + \tfrac{S}{2}]! - \log[\tfrac{1}{2}(N-N0) - \tfrac{S}{2}]! - \log N0$$

$$\simeq \cancel{\tfrac{1}{2}\log 2\pi} + (N+\tfrac{1}{2})\log N - N -$$

$$- \tfrac{1}{2}\log 2\pi - [\tfrac{1}{2}(N-N0+S) + \tfrac{1}{2}]\log[\tfrac{1}{2}(N-N0) + \tfrac{S}{2}] + + \tfrac{1}{2}(N-N0+S) -$$

$$- \tfrac{1}{2}\log 2\pi - [\tfrac{1}{2}(N-N0-S) + \tfrac{1}{2}]\log[\tfrac{1}{2}(N-N0) - \tfrac{S}{2}] + + \tfrac{1}{2}(N-N0-S) -$$

$$- \cancel{\tfrac{1}{2}\log 2\pi} - (N0+\tfrac{1}{2})\log N_0 + N_0$$

$$\simeq -\log 2\pi + (N+\tfrac{1}{2})\log N - \tfrac{1}{2}(N-N0+S+1)\log[\tfrac{1}{2}(N-N0+S)] - \tfrac{1}{2}(N-N0-S+1)\log[\tfrac{1}{2}(N-N0-S)]$$

where

$$-N + \tfrac{1}{2}(N-N0+S) + \tfrac{1}{2}(N-N0-S) + N_0 \equiv 0$$

Therefore

$$\boxed{\log g(N, N0, S) \simeq -\log 2\pi + (N+\tfrac{1}{2})\log N + (N-N0+1)\log 2 - \tfrac{1}{2}\{(N-N0+1+S)\log(N-N0+S) + (N-N0+1-S)\log(N-N0-S)\}}$$

(b) $N = 8$; $N\uparrow + N\downarrow + N0 = 8$

Since $E_{tot} = +mB \Rightarrow$ spin excess is 1

Therefore we can have only odd # spin-0 states:

$\uparrow$	0	$\downarrow$	$g(N; N\uparrow, N\downarrow, N0)$ - multiplicity
4	1	3	$8!/(4!\,1!\,3!) = 280$
3	3	2	$8!/(3!\,3!\,2!) = 560$
2	5	1	$8!/(2!\,5!\,1!) = 168$
1	7	0	$8!/(1!\,7!\,0!) = 8$
4-orbits			total $= 1016$ accessible states!

For $E_{tot} = 0 \Rightarrow$ spin excess 0 and we have only even # spin-0 states:

$\uparrow$	0	$\downarrow$	$g(N; N\uparrow, N\downarrow, N0)$ - multiplicity
4	0	4	$8!/(4!\,0!\,4!) = 70$
3	2	3	$8!/(3!\,2!\,3!) = 560$
2	4	2	$8!/(2!\,4!\,2!) = 420$
1	6	1	$8!/(1!\,6!\,1!) = 56$
0	8	0	$8!/(0!\,8!\,0!) = 1$
5-orbitals			total $= 1107$ accessible states!

Note: $3^N = 3^8 = 6561$ accessible states for the whole system!

(c) The partition function for one particle is

$$Z_1 = \exp(mB/\tau) + 1 + \exp(-mB/\tau) =$$
$$= 1 + 2\cosh(mB/\tau)$$

and the total partition function is

$$Z = Z_1^N \leftarrow \text{distinguishable particles!}$$

Therefore the free energy is

$$F = -\tau \log Z = -N\tau \log Z_1 =$$
$$= -N\tau \log[1 + 2\cosh(mB/\tau)]$$

The thermal energy is

$$U \equiv \langle \varepsilon \rangle = \tau^2 \frac{\partial}{\partial \tau}(\log Z) = N\tau^2 \frac{\partial}{\partial \tau}(\log Z_1) =$$
$$= N\cancel{\tau^2} \frac{2\sinh(mB/\tau)[-\frac{mB}{\cancel{\tau^2}}]}{[1 + 2\cosh(mB/\tau)]} =$$
$$= N(-mB) \frac{\sinh(mB/\tau)}{[\frac{1}{2} + \cosh(mB/\tau)]}$$

and the thermal energy per particle is

$$\therefore \boxed{u = \frac{U}{N} = (-mB) \frac{\sinh(mB/\tau)}{[\frac{1}{2} + \cosh(mB/\tau)]}}$$

Now we can take two limits

(I) $\frac{mB}{\tau} << 1 \leftarrow$ weak B

$$\sinh(mB/\tau) \to \left(\frac{mB}{\tau}\right)$$

$$\cosh(mB/\tau) \to 1$$

$$\therefore \boxed{u \simeq \left(-\frac{2}{3} mB\right)\left(\frac{mB}{\tau}\right) << 1}$$

(II) $\frac{mB}{\tau} >> 1 \leftarrow$ strong B

$$\sinh(mB/\tau) \to \pm \frac{1}{2} \exp\left(\frac{mB}{\tau}\right) \leftarrow \left\{\tau \gtrless 0\right\}$$

$$\cosh(mB/\tau) \to +\frac{1}{2} \exp\left(\frac{mB}{\tau}\right) >> 1$$

$$\therefore \boxed{u \simeq \mp mB} \leftarrow$$ as it should be for strong magnetic field!

$$\Downarrow$$

$$U = \mp NmB$$

The entropy is readily found to be

$$\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_V = +N\frac{\partial}{\partial \tau}\left\{\tau \log\left[1 + 2\cosh(mB/\tau)\right]\right\}$$

$$= N\log\left[1 + 2\cosh(mB/\tau)\right] \quad +$$

$$+ N\tau \frac{2\sinh(mB/\tau)}{[1+2\cosh(mB/\tau)]}\left(-\frac{mB}{\tau^2}\right) =$$

$$= N\left\{\log[1+2\cosh(mB/\tau)] - \left(\frac{mB}{\tau}\right)\frac{2\sinh(mB/\tau)}{[1+2\cosh(mB/\tau)]}\right\}$$

which is equivalent to

$$\sigma = N\left\{\log Z_1 + \frac{u}{\tau}\right\} =$$

$$= -\frac{F}{\tau} + \frac{U}{\tau} = \frac{1}{\tau}(U - F) \quad !$$

which is a check up for correctness.

Since the partition function depends only on the potential energy of the magnetic field, and there is NO extensive volume variable

$$P = -\left(\frac{\partial F}{\partial V}\right)_\tau \equiv 0$$

In order to have pressure in the system we need change of volume in the system and translational kinetic energy of the particles!

Finally we can estimate the magnetization of the system using the free energy F and the multiplicity function computed in part (a)

The entropy in the Stirling approximation is

$$\sigma(N, NO, s) \equiv \log(N, NO, s)$$

and the free energy function

$$F_L(\tau, s, B) = U(s, B) - \tau\,\sigma(N, NO, s)$$

$$= -smB + \tau\,\sigma(N, NO, s)$$

$$\therefore \left(\frac{\partial F_L}{\partial s}\right)_{\tau, B} = 0 = -mB + \tau\left(\frac{\partial \sigma}{\partial s}\right)_{N, NO}$$

$$\Rightarrow \left(\frac{mB}{\tau}\right) = \left(\frac{\partial \sigma}{\partial s}\right)_{N, NO}$$

where

$$\left(\frac{\partial \sigma}{\partial s}\right)_{N, NO} = -\frac{1}{2}\Big\{\log(N-NO+s) - \log(N-NO-s) +$$

$$+ \underbrace{\frac{(N-NO+1+s)}{(N-NO+s)}}_{(1)} - \underbrace{\frac{(N-NO+1-s)}{(N-NO-s)}}_{(2)}\Big\}$$

and disregarding the last two terms on the grounds of being of equal order and opposite signs we end up with

$$\left(\frac{\partial \sigma}{\partial s}\right)_{N,NO} \simeq -\frac{1}{2}\log\left(\frac{N-NO+s}{N-NO-s}\right)$$

which leads to

$$\frac{N-NO+\langle s\rangle}{\underbrace{N-NO}_{A}-\langle s\rangle} = \exp(-2mB/\tau)$$

or $$\frac{A+\langle s\rangle}{A-\langle s\rangle} = \exp(-2mB/\tau)$$

$$A+\langle s\rangle = A\exp(\cdots) - \langle s\rangle\exp(\cdots)$$

$$\therefore\ [1+\exp(\cdots)]\langle s\rangle = A[\exp(\cdots)-1]$$

$$\langle s\rangle = A\frac{[\exp(\cdots)-1]}{[\exp(\cdots)+1]}$$

and finally

$$\boxed{\langle s\rangle = (N-NO)\tanh(-mB/\tau)}$$

which is correct for $NO \equiv 0$!

Therefore the magnetization $\underline{M}$ is

$$\boxed{\underline{M} = \langle s\rangle m/V = \frac{(N-NO)}{V}m\tanh(-mB/\tau)}$$

(d) Generically the Boltzmann factor is given by

$$P(\varepsilon) \equiv \exp(-\varepsilon/\tau)$$

so that states with higher energy ($E > E-e$) have smaller probability for populating for one and the same temperature. But for spin-systems negative temperatures are possible for which

$$\boxed{\begin{array}{l} P(E) \equiv \exp(E/\tau) > P(E-e) \equiv \exp[(E-e)/\tau] \\ \text{when} \quad E > E-e > 0 \text{ and } \tau < 0 \end{array}}$$

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Q 4: Two-dimensional photon gas.

In 2-D the EM modes are

$$E_x = E_{x0}\sin(\omega t)\cos\left(\frac{n_x \pi x}{L}\right)\sin\left(\frac{n_y \pi y}{L}\right)$$

$$E_y = E_{y0}\sin(\omega t)\sin\left(\frac{n_x \pi x}{L}\right)\cos\left(\frac{n_y \pi y}{L}\right)$$

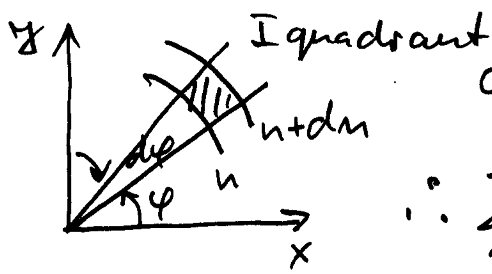
where $\vec{n}\cdot\vec{E}_0 = 0$; $\vec{n}^2 = (\vec{n}\cdot\vec{n}) = n_x^2 + n_y^2 = \dfrac{\omega^2 L^2}{c^2\pi^2}$

so that $\underbrace{\omega_n = \dfrac{|\vec{n}|c\pi}{L} = \dfrac{nc\pi}{L}}$

The average energy is going to be

$$U = \sum_n \langle \varepsilon_n \rangle = \sum_n \frac{\hbar\omega_n}{e^{\hbar\omega_n/\tau} - 1}$$

which in 2-D is converted into an integral over modes in Polar coordinates



$$dV_\varphi = n\,dn \cdot d\varphi$$

$$\therefore \sum_n \to \frac{1}{4}\int_0^\infty n\,dn \int_0^{2\pi} d\varphi = \frac{1}{4}(2\pi)\int_0 n\,dn$$

first quadrant!

$$\therefore U = \frac{\pi}{2}\int_0^\infty n\,dn\,\frac{\hbar\omega_n}{\exp(\hbar\omega_n/\tau) - 1}$$

$$\frac{\hbar\omega_n}{\tau} = \frac{\hbar n\pi c}{L\tau} \Rightarrow x = \frac{\hbar\omega_n}{\tau} = \left(\frac{\pi\hbar c}{L\tau}\right)n$$

and $$n = \left(\frac{L\tau}{\pi\hbar c}\right)x \; ; \; dn = \left(\frac{L\tau}{\pi\hbar c}\right)dx$$

$$\therefore U = \frac{\pi}{2}\int_0^\infty \left(\frac{L\tau}{\pi\hbar c}\right)^2 x\,dx \cdot \frac{\tau x}{e^x - 1} =$$

$$= \frac{\pi}{2}\left(\frac{1}{\pi}\right)^2\left(\frac{L}{\hbar c}\right)^2 \tau^3 \int_0^\infty \frac{x^2 dx}{e^x - 1} =$$

$$= \frac{1}{2\pi}\left(\frac{L}{\hbar c}\right)^2 \tau^3 \int_0^\infty \frac{x^2 dx}{e^x - 1}$$

where

$$I_2 = \int_0^\infty \frac{x^2 dx}{e^x - 1} = \int_0^\infty dx\, x^2 \frac{e^{-x}}{(1 - e^{-x})} =$$

$$= \int_0^\infty dx\, x^2 \left(e^{-x} + e^{-2x} + e^{-3x} + \dots\right)$$

and using $$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

we can integrate the above series by parts to get

$$I_2 = (2!)\left(\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \dots\right) = 2\zeta(3)$$

where (see Abramowitz & Stegun)

$$\zeta(3) = \zeta(2\cdot 1+1) = \frac{(-1)^{1+1}(2\pi)^{2\cdot 1+1}}{2(2\cdot 1+1)!}\int_0^1 B_{2\cdot 1+1}(x)\,\mathrm{cotg}(\pi x)\,dx$$

$$= \frac{(2\pi)^3}{2\cdot 3!}\int_0^1 B_3(x)\,\mathrm{cotg}(\pi x)\,dx \leftarrow \text{Difficult!}$$

and $B_3(x)$ is a Bernoulli polinomial of order 3

Unfortunately only for even # there is a closed form solutions: $\zeta(2) = \frac{\pi^2}{6}$; $\zeta(4) = \frac{\pi^4}{90}$

For odd # we have to do the integral numerically.

The result is: $\zeta(3) \simeq 1.202$

So

$$U = \frac{1}{\cancel{2}\pi}\left(\frac{L}{\hbar c}\right)^2\tau^3\cdot\cancel{2}\,\zeta(3) = \frac{1}{\pi}\zeta(3)\left(\frac{L}{\hbar c}\right)^2\tau^3$$

$$\therefore \boxed{u = \frac{U}{L^2} = \frac{U}{V} = \frac{1}{\pi}\,\frac{\zeta(3)}{(\hbar c)^2}\,\tau^3}$$

For the entropy we have

$$d\sigma = \frac{dU}{\tau} = \frac{3}{\pi}\zeta(3)\left(\frac{L}{\hbar c}\right)^2\tau\, d\tau$$

$$\therefore \boxed{\sigma = \left(\frac{3}{2\pi}\right)\zeta(3)\,V\left(\frac{\tau}{\hbar c}\right)^2}$$

and for the pressure

$$\boxed{p = -\left(\frac{\partial U}{\partial V}\right)_\sigma = +\frac{1}{\pi}\zeta(3)\frac{\tau^3}{(\hbar c)^2} \equiv +u}$$

Pressure is the negative of the energy density!
Therefore the equation of state is

$$\boxed{pV = +\frac{1}{\pi}\zeta(3)\frac{\tau^3}{(\hbar c)^2}}$$

or using

$$\boxed{p = \tau\left(\frac{\partial \sigma}{\partial V}\right)_U = \left(\frac{3}{2\pi}\right)\zeta(3)\frac{\tau^3}{(\hbar c)^2}}$$

$$\Downarrow$$

Bottom line:

$$\boxed{p \sim \tau^3 \text{ in 2-D}}$$

or $\boxed{\boxed{p = \frac{3}{2}u}}$ for 2-D photon gas

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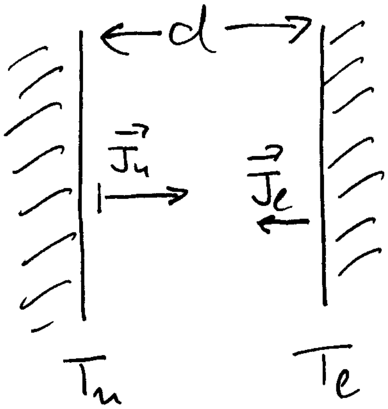
Q5: Heat shields

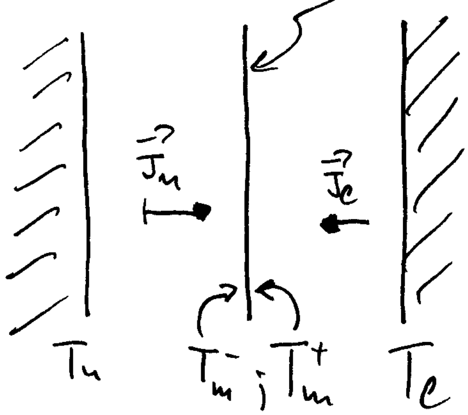
① Initial condition: $T_u > T_e$



Note: the Energy flux is actually a vector $\vec{J}_v$!

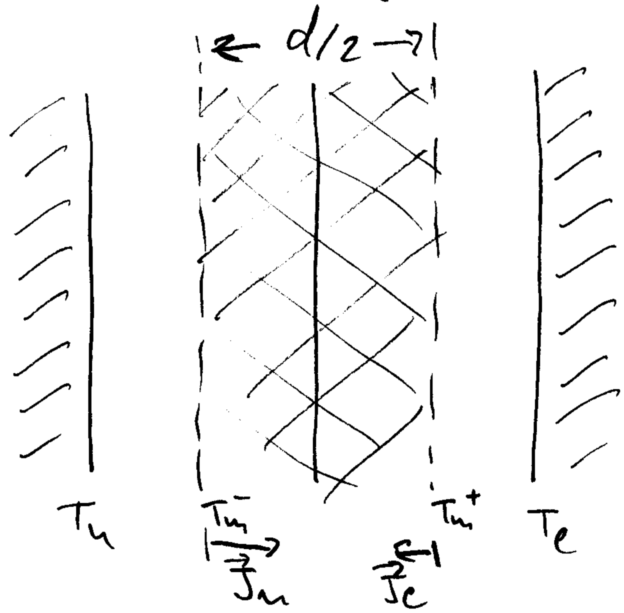
$$\vec{J}_v^{\,\circ} = \sigma_B (T_u^4 - T_e^4)\, \vec{n}$$

ⅱ ideal reflecto / absorber (Kirchhoff Law)



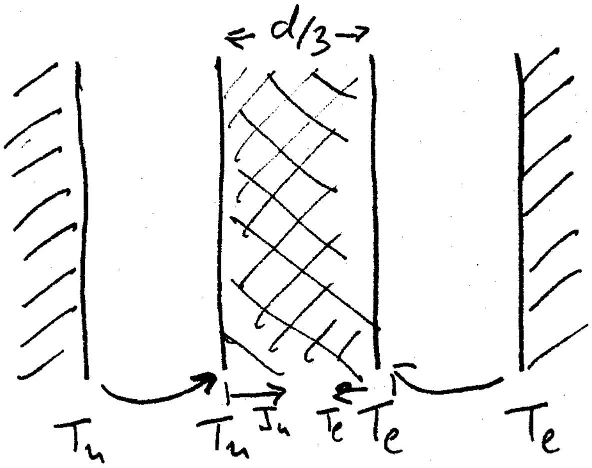
$$T_m^- = T_u \; ; \; T_m^+ = T_e$$

The reduction of the net energy flux is for purely geometrical reasons.



$$\therefore \quad |\vec{J}'| = \frac{1}{2} \sigma_B (T_u^4 - T_e^4)$$

(III)

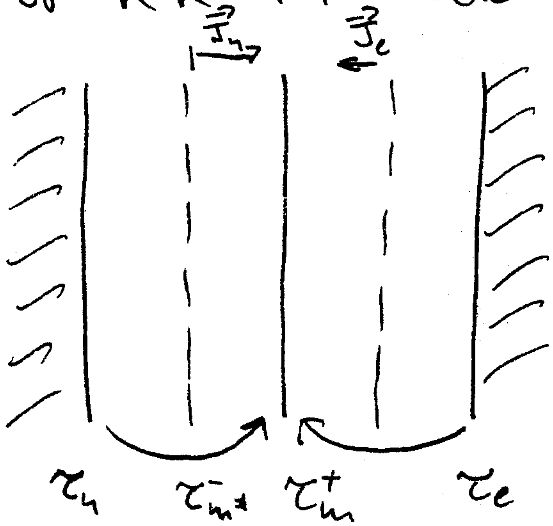


$$\Rightarrow |\vec{J}^2| = \frac{1}{3}\sigma_B (T_u^4 - T_e^4)$$

By induction you can prove that

$$|\vec{J}_v^N| = \frac{1}{(N+1)}\sigma_B (T_u^4 - T_e^4)$$

For KK 4.19 we have: $r = 1 - a$; e



$$a = 1 - r$$

$$\tau_m^- = \tau_u \; ; \; \tau_e^+ = \tau_e$$

$$|\vec{J}_u^a| = a|\vec{J}_u|$$

$$|\vec{J}_e^a| = a|\vec{J}_e|$$

$$\therefore |\vec{J}_a^1| = \frac{a}{2}\sigma_B (T_u^4 - T_e^4) = \frac{1}{2}(1-r)\sigma_B (T_u^4 - T_e^4)$$

and by induction

$$\boxed{|\vec{J}_a^N| = \frac{(1-r)}{(N+1)}\sigma_B (T_u^4 - T_e^4)}$$